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On the hyperstablity of a quadratic functional equation in commutative groups

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Abstract

Using the Brzdęk's fixed point method, we prove the hyperstablity of the functional equation

$$f(x+y+z)+f(x+y-z)+f(x-y+z)+f(x-y-z)=f(2x)+4f(y)+4f(z)$$

in the class of functions from a commutative group into a commutative complete metric group.

Key words: Hyers-Ulam stability; hyperstability; quadratic functional equation; fixed point method; complete metric space.

1 Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [33] concerning the stability of group homomorphisms. Hyers [20] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [5] for additive mappings and by Rassias [30] for linear mappings by considering an unbounded Cauchy difference. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [18] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [7, 13, 14, 16, 17, 21, 22, 26, 28, 29, 31, 32]).

We say a functional equation $\mathfrak{D}$ is *hyperstable* if any function f satisfying the equation $\mathfrak{D}$ approximately is a true solution of $\mathfrak{D}$. It seems that the first hyperstability result was published in [7] and concerned the ring homomorphisms. However, The term *hyperstability* has been used for the first time in [25]. Quite often the hyperstability is confused with superstability, which admits also bounded functions. Numerous papers on this subject have been published and we refer to [1–4, 6, 8–10, 12, 19, 25, 27].

Throughout this paper, we will denote the set of natural numbers by $\mathbb{N}$, the set of integers by $\mathbb{Z}$ and the set of real numbers by $\mathbb{R}$. Let $\mathbb{N}_+$ be the set of positive integers. By $\mathbb{N}_m$, $m \in \mathbb{N}_+$, we will denote the set of all integers greater than or equal to m. Let $\mathbb{R}_0 = [0,\infty)$ the set of nonnegative real numbers and $\mathbb{R}_+ = (0,\infty)$ the set of positive real numbers. We write B^A to mean " the family of all functions mapping from a nonempty set A into a nonempty set B ".

Definition 1.1. Let X be a nonempty set, (Y,d) be a metric space, $\varepsilon \in \mathbb{R}_0^{X^n}$ and $\mathscr{F}_1$, $\mathscr{F}_2$ be operators mapping from a nonempty set $\mathscr{D} \subset Y^X$ into Y^{X^n}. We say that the operator equation

$$\mathscr{F}_1\varphi(x_1,\ldots,x_n) = \mathscr{F}_2\varphi(x_1,\ldots,x_n), \quad (x_1,\ldots,x_n \in X) \tag{1.1}$$

is ε-hyperstable provided that every $\varphi_0 \in \mathscr{D}$ which satisfies

$$d\left(\mathscr{F}_1\varphi_0(x_1,\ldots,x_n), \mathscr{F}_2\varphi_0(x_1,\ldots,x_n)\right) \le \varepsilon(x_1,\ldots,x_n), \quad (x_1,\ldots,x_n \in X)$$

fulfills the equation (1.1).

In 2013, M. Almahalebi [3] introduced the following quadratic functional equation

$$f(x+y+z)+f(x+y-z)+f(x-y+z)+f(x-y-z) = f(2x)+4f(y)+4f(z), \tag{1.2}$$

and investigated its general solution and the generalized Hyers-Ulam-Rassias stability in Banach spaces.

In this paper, using the fixed point method derived from [11, Theorem 1], we prove the hyperstability of (1.2) in the class of functions from a commutative group into a commutative complete metric group.

Before proceeding to the main results, we state the following theorem which is useful for our purpose.

Theorem 1.2. ([11, Theorem 1]) *Let X be a nonempty set, (Y,d) a complete metric space, $f_1,\ldots,f_n\colon X \to X$ and $L_1,\ldots,L_n\colon X \to \mathbb{R}_0$ be given mappings. Let $\Lambda\colon \mathbb{R}_0^X \to \mathbb{R}_0^X$ be a linear operator defined by*

$$\Lambda\delta(x) := \sum_{i=1}^{n} L_i(x)\delta(f_i(x)), \tag{1.3}$$

for $\delta \in \mathbb{R}_0^X$ and $x \in X$. If $\mathscr{T}: Y^X \to Y^X$ is an operator satisfying the inequality

$$d\left(\mathscr{T}\xi(x), \mathscr{T}\mu(x)\right) \le \sum_{i=1}^{n} L_i(x) d\left(\xi(f_i(x)), \mu(f_i(x))\right), \quad \xi,\mu \in Y^X, x \in X,$$

and a function $\varepsilon\colon X \to \mathbb{R}_0$ and a mapping $\varphi\colon X \to Y$ satisfy

$$d\left(\mathscr{T}\varphi(x), \varphi(x)\right) \le \varepsilon(x), \quad (x \in X),$$

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} \Lambda^k \varepsilon(x) < \infty, \quad (x \in X),$$

then for every $x \in X$, the limit

$$\psi(x) := \lim_{n\to\infty} \mathscr{T}^n\varphi(x),$$

exists and the function $\psi \in Y^X$ so defined is a unique fixed point of $\mathscr{T}$ with

$$d\left(\varphi(x), \psi(x)\right) \le \varepsilon^*(x), \qquad (x \in X).$$

2 Main results

Suppose $(G,+)$ and $(H,+)$ are abelian groups, and d is a metric on H such that

(i) d is invariant with respect to $+$, that is,

$$d(u+w, v+w) = d(u,v), \qquad (u,v,w \in H);$$

(ii) (H,d) is a complete metric space.

We will denote by $Aut(G)$ the family of all automorphisms of G. Moreover, for each $u\colon G \to G$ we write $ux := u(x)$ for $x \in G$ and we define u' by $u'x := x - 2ux$ for $x \in G$.
Let

$$l(G) := \Big\{ u \in Aut(G) : u, u', 2u', (u' - 2u) \in Aut(G),$$

$$\alpha_u := \lambda(2u') + 8\lambda(u) + 2\lambda(u') + \lambda(u' - 2u) < 1 \Big\} \neq \varnothing, \tag{2.1}$$

where

$$\lambda(u) := \inf\{t \in \mathbb{R}_0 : \varepsilon(ux, uy, uz) \leq t\varepsilon(x, y, z), \quad \forall x, y, z \in G\}$$

for $u \in Aut(G)$ and $\varepsilon\colon G^3 \to \mathbb{R}_0$. The following theorem is a result concerning the hyperstability of the functional equation (1.2).

Theorem 2.1. *Let $f : G \to H$ be a mapping satisfying the inequality*

$$\begin{aligned} &d\Big(f(x+y+z), f(2x) + 4f(y) + 4f(z) - f(x+y-z) - f(x-y+z) \\ &- f(x-y-z)\Big) \leq \varepsilon(x, y, z) \end{aligned} \tag{2.2}$$

for all $x, y, z \in G$, where $\varepsilon\colon G^3 \to \mathbb{R}_0$ is an arbitrary function. Assume that there exists a nonempty subset $\mathscr{U} \subset l(G)$ such that

$$u \circ v = v \circ u, \quad \forall u, v \in \mathscr{U},$$

and

$$\begin{aligned} &\inf\{\varepsilon(u'x, ux, ux) : u \in \mathscr{U}\} = 0, \quad \forall x \in G, \\ &\sup\{\alpha_u : u \in \mathscr{U}\} < 1, \end{aligned} \tag{2.3}$$

then f is a solution of (1.2) *on G.*

Proof. Let us fix $u \in \mathscr{U}$. Replacing y and z with ux and x with $u'x$ in equation (**??**), we get

$$d\Big(f(x), f(2u'x) + 8f(ux) - 2f(u'x) - f(u'x - 2ux)\Big) \leq \varepsilon(u'x, ux, ux) := \varepsilon_u(x) \tag{2.4}$$

for all $x \in G$. We define the operators $\mathscr{T}_u\colon H^G \to H^G$ and $\Lambda_u\colon \mathbb{R}_0^G \to \mathbb{R}_0^G$ by

$$\mathscr{T}_u\xi(x) := \xi(2u'x) + 8\xi(ux) - 2\xi(u'x) - \xi\big((u' - 2u)x\big), \tag{2.5}$$

$$\Lambda_u\delta(x) := \delta(2u'x) + 8\delta(ux) + 2\delta(u'x) + \delta\big((u' - 2u)x\big)$$

for all $x \in G$, $\xi \in H^G$ and $\delta \in \mathbb{R}_0^G$. Then (2.4) becomes

$$d\Big(f(x), \mathscr{T}_u f(x)\Big) \leq \varepsilon_u(x)$$

for all $x \in G$.

The operator $\Lambda_u : \mathbb{R}_0^G \to \mathbb{R}_0^G$ has the form given by (1.3) with $s = 4$ and $L_1(x) = 1$, $L_2(x) = 8$, $L_3(x) = 2$, $L_4(x) = 1$, $f_1(x) = 2u'x$, $f_2(x) = ux$, $f_3(x) = u'x$ and $f_4(x) = (u' - 2u)x$ for all $x \in G$.
Further,

$$\begin{aligned} d\Big(\mathscr{T}_u\xi(x), \mathscr{T}_u\mu(x)\Big) =& d\Big(\xi(2u'x) + 8\xi(ux) - 2\xi(u'x) - \xi\big((u' - 2u)x\big), \\ & \mu(2u'x) + 8\mu(ux) - 2\mu(u'x) - \mu\big((u' - 2u)x\big)\Big) \\ & \leq d\Big(\xi(2u'x), \mu(2u'x)\Big) + 8d\Big(\xi(ux), \mu(ux)\Big) \\ & + 2d\Big(\xi(u'x), \mu(u'x)\Big) + d\Big(\xi\big((u' - 2u)x\big), \mu\big((u' - 2u)x\big)\Big) \end{aligned}$$

for all $x \in G$ and $\xi, \mu \in H^G$.

Note that, in view of the definition of $\lambda(u)$,

$$\varepsilon(ux, uy, uz) \leq \lambda(u)\varepsilon(x, y, z), \qquad x, y, z \in G.$$

So it is easy to show by induction on n that

$$\Lambda_u^n \varepsilon_u(x) \leq \alpha_u^n \varepsilon(u'x, ux, ux),$$

for all $x \in G$, where

$$\alpha_u = \lambda(2u') + 8\lambda(u) + 2\lambda(u') + \lambda(u' - 2u).$$

Hence

$$\varepsilon^*(x) := \sum_{k=0}^{\infty} \Lambda_u^k \varepsilon_u(x) \leq \varepsilon(u'x, ux, ux) \sum_{k=0}^{\infty} \alpha_u^k = \frac{\varepsilon(u'x, ux, ux)}{1 - \alpha_u} < \infty$$

for all $x \in G$. By Theorem 1.2, there exists a unique solution $F_u : G \to H$ of the equation

$$F_u(x) = F_u(2u'x) + 8F_u(ux) - 2F_u(u'x) - F_u\big((u' - 2u)x\big)$$

for all $x \in G$, which is a fixed point of $\mathscr{T}_u$ such that

$$d\Big(F_u(x), f(x)\Big) \leq \frac{\varepsilon(u'x, ux, ux)}{1 - \alpha_u}$$

for all $x \in G$. Moreover,

$$F_u(x) = \lim_{n \to \infty} \mathscr{T}_u^n f(x)$$

for all $x \in G$.

To prove that F_u satisfies the functional equation (1.2) on G, just prove the following inequality

$$\begin{aligned} d\Big(\mathscr{T}_u^n f(x+y+z), \mathscr{T}_u^n f(2x) + 4\mathscr{T}_u^n f(y) + 4\mathscr{T}_u^n f(z) - \mathscr{T}_u^n f(x+y-z) - \mathscr{T}_u^n f(x-y+z) \\ - \mathscr{T}_u^n f(x-y-z)\Big) \leq \alpha_u^n \varepsilon(x, y, z) \end{aligned} \tag{2.6}$$

for all $x, y, z \in G$, and $n \in \mathbb{N}$.

Indeed, if $n = 0$ then (2.6) is simply (2.2). So, take $n \in \mathbb{N}_+$ and suppose that (2.6) holds for n and $x, y \in G$. Then, by using (2.5) and the triangle inequality, we have

$$
\begin{aligned}
& d\Big(\mathscr{T}_u^{n+1}f(x+y+z), \mathscr{T}_u^{n+1}f(2x)+4\mathscr{T}_u^{n+1}f(y)+4\mathscr{T}_u^{n+1}f(z)-\mathscr{T}_u^{n+1}f(x+y-z) \\
& -\mathscr{T}_u^{n+1}f(x-y+z)-\mathscr{T}_u^{n+1}f(x-y-z)\Big) \\
& = d\Big(\mathscr{T}_u^n f\big(2u'(x+y+z)\big)+8\mathscr{T}_u^n f\big(u(x+y+z)\big)-2\mathscr{T}_u^n f\big(u'(x+y+z)\big) \\
& -\mathscr{T}_u^n f\big((u'-2u)(x+y+z)\big), \mathscr{T}_u^n f\big(4u'x\big)+8\mathscr{T}_u^n f\big(2ux)\big)-2\mathscr{T}_u^n f\big(2u'x)\big)-\mathscr{T}_u^n f\big(2(u'-2u)x)\big) \\
& +4\mathscr{T}_u^n f\big(2u'y\big)+32\mathscr{T}_u^n f\big(uy)\big)-8\mathscr{T}_u^n f\big(u'y)\big)-4\mathscr{T}_u^n f\big((u'-2u)y)\big) \\
& +4\mathscr{T}_u^n f\big(2u'z\big)+32\mathscr{T}_u^n f\big(uz)\big)-8\mathscr{T}_u^n f\big(u'z)\big)-4\mathscr{T}_u^n f\big((u'-2u)z)\big) \\
& -\mathscr{T}_u^n f\big(2u'(x+y-z)\big)+8\mathscr{T}_u^n f\big(u(x+y-z)\big)-2\mathscr{T}_u^n f\big(u'(x+y-z)\big) \\
& -\mathscr{T}_u^n f\big((u'-2u)(x+y-z)\big)\Big) \\
& \leq d\Big(\mathscr{T}_u^n f\big(2u'(x+y+z)\big), \mathscr{T}_u^n f(4u'x)+4\mathscr{T}_u^n f(2u'y)+4\mathscr{T}_u^n f(2u'z)-\mathscr{T}_u^n f\big(2u'(x+y-z)\big) \\
& -\mathscr{T}_u^n f\big(2u'(x-y+z)\big)-\mathscr{T}_u^n f\Big(2u'(x-y-z)\big)\Big) \\
& +8d\Big(\mathscr{T}_u^n f\big(u(x+y+z)\big), \mathscr{T}_u^n f(2ux)+4\mathscr{T}_u^n f(uy)+4\mathscr{T}_u^n f(uz)-\mathscr{T}_u^n f\big(u(x+y-z)\big) \\
& -\mathscr{T}_u^n f\big(u(x-y+z)\big)-\mathscr{T}_u^n f\big(u(x-y-z)\big)\Big) \\
& +2d\Big(\mathscr{T}_u^n f\big(u'(x+y+z)\big), \mathscr{T}_u^n f(2u'x)+4\mathscr{T}_u^n f(u'y)+4\mathscr{T}_u^n f(u'z)-\mathscr{T}_u^n f\big(u'(x+y-z)\big) \\
& -\mathscr{T}_u^n f\big(u'(x-y+z)\big)-\mathscr{T}_u^n f\big(u'(x-y-z)\big)\Big) \\
& +d\Big(\mathscr{T}_u^n f\big((u'-2u)(x+y+z)\big), \mathscr{T}_u^n f(2(u'-2u)x)+4\mathscr{T}_u^n f((u'-2u)y)+4\mathscr{T}_u^n f((u'-2u)z) \\
& -\mathscr{T}_u^n f\big((u'-2u)(x+y-z)\big)-\mathscr{T}_u^n f\big((u'-2u)(x-y+z)\big)-\mathscr{T}_u^n f\big((u'-2u)(x-y-z)\big)\Big) \\
& = \alpha_u^{n+1}\varepsilon(x,y,z).
\end{aligned}
$$

By induction, we have shown that (2.6) holds for all $x,y,z \in G$. Letting $n \to \infty$ in (2.6), we get

$$
F_u(x+y+z) = F_u(2x)+4F_u(y)+4F_u(z)-F_u(x+y-z)-F_u(x-y+z)-F_u(x-y-z)
$$

for all $x,y,z \in G$. Thus, we have proved that for every $u \in \mathscr{U}$ there exists a function $F_u : G \to H$ which is a solution of the functional equation (1.2) on G and satisfies

$$
d\Big(f(x), F_u(x)\Big) \leq \frac{\varepsilon(u'x, ux, ux)}{1-\alpha_u}
$$

for all $x \in G$. By (2.3), we get

$$
\begin{aligned}
d\Big(f(x), F_u(x)\Big) \quad & \leq \frac{\inf_{u\in\mathscr{U}} \varepsilon(u'x, ux, ux)}{1-\sup_{u\in\mathscr{U}} \alpha_u} \\
& = 0
\end{aligned}
$$

for all $x \in G$. This means that $F_u(x) = f(x)$ for all $x \in G$ and $u \in \mathscr{U}$, and hence

$$
f(x+y+z) = f(2x)+4f(y)+4f(z)-f(x+y-z)-f(x-y+z)-f(x-y-z)
$$

for all $x,y,z \in G$, which implies that f satisfies the functional equation (1.2) on G. □

In the next theorem, we will study the hyperstablity of the functional equation (1.2) on G without 0 the neutral element, because of the reason that one can easily deduce some applications.

Theorem 2.2. *Let $f : G \to H$ be a mapping satisfying the inequality*

$$d\Big(f(x+y+z), f(2x)+4f(y)+4f(z)-f(x+y-z)-f(x-y+z)$$
$$-f(x-y-z)\Big) \leq \varepsilon(x,y,z)$$

for all $x,y,z \in G\backslash\{0\}$, where 0 is the neutral element of the group $(G,+)$ and $\varepsilon : (G\backslash\{0\})^3 \to \mathbb{R}_0$ is an arbitrary function. Assume that there exists a nonempty subset $\mathscr{U} \subset l(G)$ such that

$$u \circ v = v \circ u \qquad (u,v \in \mathscr{U}),$$

and

$$\begin{aligned} &\inf\{\varepsilon(u'x, ux, ux) : u \in \mathscr{U}\} = 0, \quad \forall x \in G\backslash\{0\}, \\ &\sup\{\alpha_u : u \in \mathscr{U}\} < 1, \end{aligned} \tag{2.7}$$

then f is a solution of the functional equation (1.2) *on $G\backslash\{0\}$.*

Proof. The proof is the same as in the proof of Theorem 2.1. □

3 Applications

From Theorem 2.2, we can obtain the following corollaries as natural results.

Corollary 3.1. *Let E and F be a normed space and a Banach space, respectively. Assume that X is a subgroup of the group $(E,+)$, $p<0$, $q<0$, $r<0$ and $\theta \geq 0$. If $f : X \to F$ satisfies*

$$\begin{aligned} \big\| f(x+y+z)) + f(x+y-z) + f(x-y+z) + f(x-y-z) - f(2x) - 4f(y) - 4f(z) \big\| \\ \leq \theta\Big(\|x\|^p + \|y\|^q + \|z\|^r \Big) \end{aligned} \tag{3.1}$$

for all $x,y,z \in X\backslash\{0\}$, then f satisfies the functional equation (1.2) *on $X\backslash\{0\}$.*

Proof. The proof follows from Theorem 2.2 by taking

$$\varepsilon(x,y,z) = \theta\Big(\|x\|^p + \|y\|^q + \|z\|^r \Big), \quad x,y,z \in X\backslash\{0\},$$

with some real numbers $\theta \geq 0$, $p<0$, $q<0$, $r<0$ and $d(x,y) = \|x-y\|$. For each $m \in \mathbb{N}_+$ define $u_m \colon X\backslash\{0\} \to X\backslash\{0\}$ by $u_m x := -mx$ and $u'_m \colon X\backslash\{0\} \to X\backslash\{0\}$ by $u'_m x := (1+2m)x$. Then

$$\begin{aligned} \varepsilon(u_m x, u_k y, u_l z) \qquad &= \varepsilon(-mx, -ky, -lz) \\ &= \theta\Big(\|-mx\|^p + \|-ky\|^q + \|-lz\|^r \Big) \\ &= \theta m^p \|x\|^p + \theta k^q \|y\|^q + \theta l^r \|z\|^r \\ &\leq (m^p + k^q + l^r)\, \theta\Big(\|x\|^p + \|y\|^q + \|z\|^r \Big) \\ &= (m^p + k^q + l^r)\, \varepsilon(x,y,z) \end{aligned}$$

for all $x,y,z \in X \setminus \{0\}$, $k,l,m \in \mathbb{N}_+$. Hence

$$\begin{aligned} \lim_{m\to\infty} \varepsilon(u'_m x, u_m y, u_m z) \qquad &\leq \lim_{m\to\infty} \Big((1+2m)^p + m^q + m^r \Big) \varepsilon(x,y,z) \\ &= 0 \end{aligned}$$

for all $x,y,z \in X\backslash\{0\}$. Then (2.7) is valid with $\lambda(u_m) = m^p + m^q + m^r$ for $m \in \mathbb{N}_+$, and there exists $1 < \mathscr{M} < m$ such that

$$\begin{aligned} \lambda(2u'_m) + 8\lambda(u_m) + 2\lambda(u'_m) + \lambda(u'_m - 2u_m) = \big(2^p(1+2m)^p + 2^q(1+2m)^q + 2^r(1+2m)^r\big) \\ +8\big(m^p + m^q + m^r\big) + 2\big((1+2m)^p + (1+2m)^q \\ +(1+2m)^r\big) + (1+4m)^p + (1+4m)^q + (1+4m)^r < 1. \end{aligned}$$

So it easily seen that (2.1) is fulfilled with

$$\mathscr{U} := \{u_m \in Aut\ X : m \in \mathbb{N}_{\mathscr{M}}\}.$$

Therefore, by Theorem 2.2, every $f : X \to F$ satisfying (3.1) is a solution of the functional equation (1.2) on $X\backslash\{0\}$. $\square$

Corollary 3.2. *Let E and F be a normed space and a Banach space, respectively. Assume that X is a subgroup of the group $(E,+)$, $p,q,r \in \mathbb{R}$, $p+q+r < 0$ and $\theta \geq 0$. If $f : X \to F$ satisfies*

$$\left\| f(x+y+z)) + f(x+y-z) + f(x-y+z) + f(x-y-z) - f(2x) - 4f(y) - 4f(z) \right\| \\ \leq \theta \|x\|^p \|y\|^q \|z\|^r$$

for all $x,y,z \in X\backslash\{0\}$, then f satisfies the functional equation (1.2) *on $X\backslash\{0\}$.*

Proof. It is easily seen that the function ε given by

$$\varepsilon(x,y,z) = \theta \|x\|^p \|y\|^q \|z\|^r \quad x,y,z \in X\backslash\{0\},$$

satisfies (2.7), since

$$\begin{aligned} \varepsilon(mx,ky,lz) \quad &= \theta \|mx\|^p \|ky\|^q \|lz\|^r \\ &= \theta |m|^p |k|^q |l|^r \|x\|^p \|y\|^q \|z\|^r \\ &= |m|^p |k|^q |l|^r \varepsilon(x,y,z) \end{aligned}$$

for all $x,y,z \in X\backslash\{0\}$, $k,l,m \in \mathbb{Z}$, and $klm \neq 0$.

The remainder of the proof is similar to the proof of Corollary 3.1. $\square$

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