A generalized Poincaré inequality for Jacobi operators

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Abstract. The aim of this note is to study the Jacobi semigroup \( (P^\alpha_\mu)_t \), \( \alpha, \beta > -1 \), generated by the operator \( L^\alpha_\mu f(x) := (1-x^2)f'' + [(\beta - \alpha) - (\alpha + \beta + 2)x]f' \) acting on the Hilbert space \( L^2([-1,1], \mu) \) with respect to the normalized Jacobi probability measure \( \mu_\mu(dx) = C_\alpha_\beta (1-x^\alpha)(1+x^\beta)dx \). By means of a method involving essentially a commutation property between the semigroup and the derivation, we establish a family of inequalities with the Poincaré inequality as a particular case.

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1 Introduction

The celebrated logarithmic Sobolev inequality of Gross [3] expresses that for all smooth functions \( f \) on \( \mathbb{R}^d \),

\[
\int_{\mathbb{R}^d} f^2 \log f^2 \, d\gamma_d - \left( \int_{\mathbb{R}^d} f^2 \, d\gamma_d \right) \log \left( \int_{\mathbb{R}^d} f^2 \, d\gamma_d \right) \leq 2 \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma_d,
\]

(1)

where \( |\nabla f| \) is the length of the usual gradient of \( f \) and \( \gamma_d \) is the \( d \)-dimensional standard Gaussian measure given by \( \gamma_d(dx) := \left( \frac{\sqrt{2\pi}}{\sqrt{d}} \right)^{-d} \exp(-|x|^2/2)dx \).

Inequality (1) is in fact a reinforced form of the Poincaré inequality (also known as spectral gap inequality) [3] :

\[
\int_{\mathbb{R}^d} f^2 \, d\gamma_d - \left( \int_{\mathbb{R}^d} f \, d\gamma_d \right)^2 \leq \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma_d,
\]

(2)

for all smooth functions \( f \) on \( \mathbb{R}^d \). In the paper [1], W. Beckner derived a family of generalized Poincaré inequalities that yield a sharp interpolation between the two inequalities (1) and (2) :

\[
\int_{\mathbb{R}^d} f^2 \, d\gamma_d - \int_{\mathbb{R}^d} e^{tH_d} f \, d\gamma_d \leq (1 - e^{-2t}) \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma_d, \quad \text{for all } t \geq 0.
\]

(3)

where \( H_d \) is the classical Ornstein-Uhlenbeck operator given by :

\[
H_d f(x) := \Delta f(x) - \langle x, \nabla f(x) \rangle, \quad x \in \mathbb{R}^d
\]

Here \( \Delta \) and \( \langle \cdot , \cdot \rangle \), respectively stand for the Laplacian operator and the scalar product on \( \mathbb{R}^d \).

In the Gaussian context, C. Houdré and A. Kagan [4] extended the inequality (2) in the following sense : for all smooth functions \( f \) on \( \mathbb{R}^d \) with square integrable iterated gradients, then for \( n = 1, 2, \ldots , \)

\[
\sum_{k=1}^{2n} \frac{(-1)^{k+1}}{k!} \int_{\mathbb{R}^d} |\nabla^k f|^2 \, d\gamma_d \leq \int_{\mathbb{R}^d} f^2 \, d\gamma_d - \left( \int_{\mathbb{R}^d} f \, d\gamma_d \right)^2 \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k+1}}{k!} \int_{\mathbb{R}^d} |\nabla^k f|^2 \, d\gamma_d,
\]

(4)

where the \( \nabla^k \) are the iterated gradients and where \( |\cdot| \) is the Euclidian norm in the corresponding spaces \( \mathbb{R}^{kd} \). These inequalities are tightly related to the Ornstein-Ulenbeck operator. Similar researches on this kind of inequalities for general probability measures generated by diffusions have been done by many authors (see [5] for instance).

For the Jacobi operator, defined on the interval \([ -1, 1 ]\) by

\[
(1 - x^2)f'' + [(\beta - \alpha) - (\alpha + \beta + 2)x]f', \quad \alpha, \beta > -1,
\]

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the Poincaré inequality is simply the estimate: for all smooth functions $f$ on $I$,  
\[ \int_{-1}^{1} f^2 \mu_{\alpha,\beta} - \left( \int_{-1}^{1} f \mu_{\alpha,\beta} \right)^2 \leq \frac{1}{\alpha + \beta + 2} \int_{-1}^{1} (1-x^2)f''(x)\mu_{\alpha,\beta} \]  
(5)

where $\mu_{\alpha,\beta}(dx) = C_{\alpha,\beta}(1-x)\alpha(1+x)\beta dx$ denotes the normalized Jacobi probability measure on $I$. In [2], E. Fontenas was interested in the minimization of the Sobolev constant and the logarithmic Sobolev constant for the Jacobi operator. These constants are different from the first non-negative eigenvalue associated to these operators.

The objective of the present note is to analyze the heat semigroup for the Jacobi operator. The structure of the paper is as follows: in the next section we recall briefly some needed spectral properties of the Jacobi operator. Following the idea of W. Beckner [1], and C. Houdré and A. Kagan [4] in the Gaussian case, we propose in section 3 an extension of the usual Poincaré inequality for the Jacobi probability measure on ]$-1,1[$ (see Theorem 3.1 below).

2 The heat semigroup for the Jacobi operator

In this section, we bring out some preliminary results on Jacobi differential operators. Let $I := ]-1,1[$. For fixed parameters $\alpha, \beta > -1$, we denote the Jacobi operator $L^{\alpha,\beta}$ acting on $C^\infty(I)$ by  
\[ L^{\alpha,\beta} f(x) := (1-x^2)f'' + [(\beta - \alpha) - (\alpha + \beta + 2)x]f', \ x \in I. \]

Their name comes from the famous Jacobi polynomials $\{Q_k^{\alpha,\beta} : k = 0, 1, 2, \ldots\}$ defined by  
\[ Q_k^{\alpha,\beta}(x) := \frac{(-1)^k}{k!} \sum_{i=0}^{k} \frac{(-k)_i (k+\alpha+\beta+1)_i}{i!(\alpha+1)_i} \left( 1 - \frac{x}{2} \right)^i, \ x \in I, \ k = 0, 1, 2, \ldots, \]

where $(a)_i = a(a+1)\ldots(a+i-1)$, which are eigenvectors for $L^{\alpha,\beta}$ (see, for example, [6])  
\[ L^{\alpha,\beta}(Q_k^{\alpha,\beta}) = -\lambda_k^{\alpha,\beta}Q_k^{\alpha,\beta}, \quad \text{where we set } \lambda_k^{\alpha,\beta} = k(k+\alpha+\beta+1), \ k = 0, 1, 2, \ldots. \]

We also introduce the Jacobi probability measure $\mu_{\alpha,\beta}$ defined on $I$ by  
\[ c_{\alpha,\beta}(1-x)\alpha(1+x)\beta dx = \frac{\Gamma(\alpha + \beta + 2)}{2^{\alpha + \beta + 1} \Gamma(\alpha + 1) \Gamma(\beta + 1)}(1-x)^\alpha(1+x)^\beta dx, \]

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where \( \Gamma \) is the usual gamma function:

\[
\Gamma(y) = \int_0^{+\infty} t^{y-1}e^{-t}dt, \quad y > 0.
\]

In fact, the Jacobi distribution \( \mu_{\alpha,\beta} \) is symmetrizing for the operator \( L^{\alpha,\beta} \) and the sequence \( \left(-\lambda_k^{\alpha,\beta}, \text{Vect}(Q_k^{\alpha,\beta})\right)_{k=0,1,2,...} \) forms the spectral decomposition of the minimal self-adjoint extension of this operator on the Hilbert space \( L^2(I, \mu_{\alpha,\beta}) \).

Following the usual notation, we write \( \langle f \rangle_{\alpha,\beta} \) for the integral of a function \( f \) defined on \( I \) and integrable with respect to the measure \( \mu_{\alpha,\beta} \). The inner product of two functions \( f \) and \( g \) in the space \( L^2(I, \mu_{\alpha,\beta}) \) is designated by \( \langle f, g \rangle_{\alpha,\beta} \), or simply \( \langle fg \rangle_{\alpha,\beta} \).

Let \( f \) and \( g \) be functions in \( C^2(I) \). According to an integration by parts on the second derivative, it is easy to establish the symmetry and dissipativity formulas:

\[
\langle -L^{\alpha,\beta}f, g \rangle_{\alpha,\beta} = \langle f, -L^{\alpha,\beta}g \rangle_{\alpha,\beta} = \langle (1 - x^2)f'(x)g'(x) \rangle_{\alpha,\beta}.
\]

(6)

The diffusion semigroup \( (P_t^{\alpha,\beta})_{t \geq 0} \) generated by the operator \( L^{\alpha,\beta} \) can be expressed, for any \( t \geq 0 \), by

\[
P_t^{\alpha,\beta}f := \exp(tL^{\alpha,\beta})f = \sum_{k \geq 0} e^{-t\lambda_k^{\alpha,\beta}} \hat{f}(k)Q_k^{\alpha,\beta},
\]

(7)

where \( f = \sum_{k \geq 0} \hat{f}(k)Q_k^{\alpha,\beta} \) and \( \hat{f}(k) \) is the Jacobi Fourier coefficient of \( f \) given by

\[
\hat{f}(k) := \frac{\langle f, Q_k^{\alpha,\beta} \rangle_{\alpha,\beta}}{\|Q_k^{\alpha,\beta}\|_2}, \quad k = 0, 1, 2, \ldots,
\]

\( \| \cdot \|_2 \) is the \( L^2(I, \mu_{\alpha,\beta}) \)-norm with respect to \( \mu_{\alpha,\beta} \). Thus, \( (P_t^{\alpha,\beta})_{t \geq 0} \) defines a Markovian semigroup of positive contractions in all \( L^p(I, \mu_{\alpha,\beta}) \) \((1 \leq p \leq +\infty)\) with \( \mu_{\alpha,\beta} \) as the invariant and symmetric measure:

\[
\langle P_t^{\alpha,\beta}f, g \rangle_{\alpha,\beta} = \langle f, P_t^{\alpha,\beta}g \rangle_{\alpha,\beta}, \quad f, g \in L^2(I, \mu_{\alpha,\beta}),
\]

known as the Jacobi semigroup with respect to a Jacobi probability measure \( \mu_{\alpha,\beta} \).

To obtain an integral representation of \( P_t^{\alpha,\beta} \), we insert the integral defining \( \hat{f}(k) \) into (7) and then, using Fubini’s theorem, we interchange the order of summation and integration. The result is

\[
P_t^{\alpha,\beta}f(x) = \langle K_t^{\alpha,\beta}(x, \cdot)f(\cdot) \rangle_{\alpha,\beta}, \quad f \in L^1(I, \mu_{\alpha,\beta}), \quad x \in I,
\]

(8)
where
\[ K^{\alpha,\beta}_t(x, y) = \sum_{k \geq 0} e^{-t\lambda_k^{\alpha,\beta}} \frac{Q_k^{\alpha,\beta}(x)Q_k^{\alpha,\beta}(y)}{\|Q_k^{\alpha,\beta}\|_2^2}, \quad x, y \in I. \]

The above kernel is smooth for \( x, y \in I, \ t \geq 0 \), and the integral in (8) is absolutely convergent. The Poincaré inequality (5) previously cited in the introduction was the key point to obtain the notion of \( L^2(I, \mu_{\alpha,\beta}) \)-ergodicity of the semigroup \((P_{t}^{\alpha,\beta})_{t \geq 0}\) : Indeed, since \( \mu_{\alpha,\beta} \) is invariant, the inequality (5) applied to \( P_{t}^{\alpha,\beta}f, \ f \in C^\infty(I) \), yields
\[ -\frac{d}{dt} \langle (P_{t}^{\alpha,\beta}f)^2 \rangle_{\alpha,\beta} \geq 2\lambda_1^{\alpha,\beta} \langle (P_{t}^{\alpha,\beta}f - \langle f \rangle_{\alpha,\beta})^2 \rangle_{\alpha,\beta}, \quad t \geq 0. \]

Then
\[ [0, \infty \ni t \mapsto e^{2\lambda_1^{\alpha,\beta}t} \langle (P_{t}^{\alpha,\beta}f - \langle f \rangle_{\alpha,\beta})^2 \rangle_{\alpha,\beta} \]

is decreasing. We deduce the ergodicity property:
\[ \|P_{t}^{\alpha,\beta}f - \langle f \rangle_{\alpha,\beta}\|_2^2 \leq e^{-2\lambda_1^{\alpha,\beta}t} \|f - \langle f \rangle_{\alpha,\beta}\|_2^2, \quad t \geq 0. \]

Before pursuing, we bring a heuristic identity that will be very useful in our development.

**Lemma 2.1 (The commutation property)** Let \( f : I \rightarrow \mathbb{R} \) be of class \( C^\infty(I) \), then for all \( \alpha, \beta > -1 \) and all \( t \geq 0 \),
\[ \left( P_{t}^{\alpha,\beta} f \right)^{(k)} = e^{-t\lambda_k^{\alpha,\beta}} P_{t}^{\alpha+k,\beta+k} f^{(k)}, \quad k = 0, 1, 2, \ldots \]
where \((P_{t}^{\alpha+k,\beta+k})_{t \geq 0}\) designates the heat semigroup generated by the operator \( L^{\alpha+k,\beta+k} \).

**Proof of Lemma 2.1** The formula (9) is an immediate consequence of the commutation relation between the action of the operator \( L^{\alpha,\beta} \) and the derivation:
\[ \left( L^{\alpha,\beta} f \right)^{(k)} = L^{\alpha+k,\beta+k} f^{(k)} - \lambda_k^{\alpha,\beta} f^{(k)}, \quad k = 0, 1, 2, \ldots, \]
at any point \( x \) in \( I \). We let \( t \geq 0, \ f \in C^\infty(I) \), be fixed and we consider the function \( \Psi \) defined on the interval \([0,t]\) by
\[ \Psi(s) := e^{-(t-s)\lambda_k^{\alpha,\beta}} P_{t-s}^{\alpha+k,\beta+k} \left( P_{s}^{\alpha,\beta} f \right)^{(k)}, \quad 0 \leq s \leq t. \]
$\Psi$ is constant, since its derivative is

$$\Psi'(s) = e^{-(t-s)\lambda_{k}^{\alpha,\beta}} P_{t-s}^{\alpha+k,\beta+k} \left( \lambda_{k}^{\alpha,\beta} g_{s}^{(k)} - \left( L_{s}^{\alpha,\beta} g_{s}^{(k)} \right)^{(k)} + L_{s+k,\beta+k}^{\alpha,\beta} g_{s}^{(k)} \right) = 0,$$

where $g_{s} = P_{s}^{\alpha,\beta} f$. In this last equality, we made use of the commutation relation (10). In particular, $\Psi(t) = \Psi(0)$, giving the identity (9). This ends the proof of Lemma 2.1.

## 3 Extention of the Poincaré inequality

Following a strategy of W. Beckner [1], and C. Houdré and A. Kagan [4] in the Gaussian case, we use some basic properties of the diffusion semigroup $(P_{t}^{\alpha,\beta})_{t \geq 0}$ to study integral inequalities related to the Jacobi operator $L_{s}^{\alpha,\beta}$. The commutation relation between the semigroup and the derivation (9) appears as a main tool in this work.

From now on, we adapt the following notation: For a sufficiently smooth function $f$, we set for any $\tau \geq 0$,

$$\text{Var}_{\tau}(f) := \langle f^{2} \rangle_{\alpha,\beta} - \langle (P_{\tau}^{\alpha,\beta} f)^{2} \rangle_{\alpha,\beta}, \quad \tau \geq 0,$$

and

$$\text{Var}_{\infty}(f) := \text{Var}(f) := \langle f^{2} \rangle_{\alpha,\beta} - \langle f \rangle_{\alpha,\beta}^{2}.$$

$\text{Var}_{\tau}(f)$ is non-negative, since the function $x \mapsto x^{2}$ is convex and $(P_{t}^{\alpha,\beta})_{t \geq 0}$ is an order preserving contractive semigroup. The Poincaré inequality is simply the estimate

$$\text{Var}(f) \leq \frac{1}{\lambda_{1}^{\alpha,\beta}} \langle (1 - x^{2}) f^{2}(x) \rangle_{\alpha,\beta}. \quad (11)$$

Consider at present the inequality

$$\text{Var}_{\tau}(f) \leq \frac{1 - e^{-2\tau \lambda_{1}^{\alpha,\beta}}}{\lambda_{1}^{\alpha,\beta}} \langle (1 - x^{2}) f^{2}(x) \rangle_{\alpha,\beta}, \quad \tau \geq 0, \quad (12)$$

whose proof is contained in the one of the main Theorem below. Inequality (12) is sharp and the equality is achieved for $f$ of the form $x \mapsto x + \frac{\alpha - \beta}{\alpha + \beta + 2}$. Now, as $\tau \to +\infty$, then $P_{\tau}^{\alpha,\beta} f \to \langle f \rangle_{\alpha,\beta}$ and (12) recovers the Poincaré inequality (11). In fact, Inequality (12) is just a special case of a family of more general inequalities. More precisely, we have:
Théorème 3.1 (Extension of the Poincaré inequality) Let \( f : I \rightarrow \mathbb{R} \) be of class \( C^\infty(I) \) and let \( \tau \geq 0 \). Then, for all \( \alpha, \beta > -1 \) and all integer \( n \geq 1 \),
\[
\sum_{k=1}^{2n} \frac{(-1)^{k+1} \Omega_k(\tau)}{\pi_k} \langle (1-x^2)^k f^{(k)} \rangle_{\alpha,\beta} \leq \text{Var}_\tau(f) \leq \sum_{k=1}^{2n-1} \frac{(-1)^{k+1} \Omega_k(\tau)}{\pi_k} \langle (1-x^2)^k f^{(k)} \rangle_{\alpha,\beta}
\]
where
\[
\pi_0 = \Omega_0(\tau) = 1, \quad \pi_n = \prod_{i=1}^{n} \lambda_i^{\alpha,\beta} \quad \text{and} \quad \Omega_n(\tau) = 1 - \sum_{i=1}^{n} e^{-2\tau \lambda_i^{\alpha,\beta}} \prod_{j \neq i}^{n} \left( \frac{\lambda_j^{\alpha,\beta}}{\lambda_i^{\alpha,\beta} - \lambda_j^{\alpha,\beta}} \right).
\]

This theorem is a consequence of the following proposition that expresses a sort of the Taylor formula for \( \langle (P_t^{\alpha,\beta} f)^2 \rangle_{\alpha,\beta} \).

Proposition 3.2 With the hypotheses and notation of Theorem 3.1 we have, for all \( \tau \geq 0 \), all \( \alpha, \beta > -1 \) and all integer \( n \geq 1 \),
\[
\langle (P_t^{\alpha,\beta} f)^2 \rangle_{\alpha,\beta} = \sum_{k=0}^{n-1} \frac{(-1)^k}{\pi_k} \Omega_k(\tau) \langle (1-x^2)^k f^{(k)} \rangle_{\alpha,\beta} + \frac{(-1)^n}{\pi_n-1} \int_0^\tau 2 e^{-2t \lambda_n^{\alpha,\beta}} \Omega_{n-1}(\tau-t) \langle (1-x^2)^n (P_t^{\alpha+n,\beta+n}(f^{(n)}))^2 \rangle_{\alpha,\beta} \, dt
\]

Proof of Proposition 3.2: The proof is done by induction. For \( n = 1 \), using the dissipativity formula (6) and the commutation relation (9), one may write,
\[
\langle (P_t^{\alpha,\beta} f)^2 \rangle_{\alpha,\beta} - \langle f^2 \rangle_{\alpha,\beta} = \int_0^\tau \frac{d}{dt} \left[ \langle (P_t^{\alpha,\beta} f)^2 \rangle_{\alpha,\beta} \right] \, dt = 2 \int_0^\tau \langle [L^{\alpha,\beta} P_t^{\alpha,\beta} f, P_t^{\alpha,\beta} f] \rangle_{\alpha,\beta} \, dt = -2 \int_0^\tau e^{-2t \lambda_n^{\alpha,\beta}} \langle (1-x^2)^n (P_t^{\alpha+n,\beta+n}(f^{(n)}))^2 \rangle_{\alpha,\beta} \, dt,
\]
We find (11) for \( n = 1 \). Assume that (14) is true for the integer \( n \geq 1 \). Let us show its validity for \( n+1 \). Using an integration by parts over the time variable \( t \), the integral in (14) becomes:
\[
- \frac{1}{\lambda_n^{\alpha,\beta}} e^{-2t \lambda_n^{\alpha,\beta}} \Omega_n(\tau-t) \bigg|_0^\tau + \frac{1}{\lambda_n^{\alpha,\beta}} \int_0^\tau e^{-2t \lambda_n^{\alpha,\beta}} \Omega_n(\tau-t) \frac{d}{dt} \langle (1-x^2)^n (P_t^{\alpha+n,\beta+n}(f^{(n)}))^2 \rangle_{\alpha,\beta} \, dt
\]
\[
= \frac{\Omega_n(\tau)}{\lambda_n^{\alpha,\beta}} + \frac{1}{\lambda_n^{\alpha,\beta}} \int_0^\tau e^{-2t \lambda_n^{\alpha,\beta}} \Omega_n(\tau-t) \frac{d}{dt} \langle (1-x^2)^n (P_t^{\alpha+n,\beta+n}(f^{(n)}))^2 \rangle_{\alpha,\beta} \, dt,
\]
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where we used in the last equality the fact that \( \Omega_n(0) = 0 \).

Invoking again formulas (6) and (9) of the operator \( L^\alpha + \nu^+ \), we get

\[
\frac{d}{dt} \langle (1 - x^2)^n (P_t^{\alpha + \nu^+} f(n))^2 \rangle_{\alpha, \beta} = 2 \langle (1 - x^2)^n L^{\alpha + \nu^+} P_t^{\alpha + \nu^+} f(n), P_t^{\alpha + \nu^+} f(n) \rangle_{\alpha, \beta}
\]

\[
= 2 \frac{C_{\alpha, \beta}}{C_{\alpha + \nu^+, \beta + \nu^+}} \langle L^{\alpha + \nu^+} P_t^{\alpha + \nu^+} f(n), P_t^{\alpha + \nu^+} f(n) \rangle_{\alpha + \nu^+, \beta + \nu^+}
\]

\[
= -2e^{-2t\lambda_n^{\alpha + \nu^+}} \langle (1 - x^2)^{n+1} (P_t^{\alpha + \nu^+ + 1} f(n+1))^2 \rangle_{\alpha, \beta},
\]

which enables us to conclude our induction reasoning. The proof of Proposition 3.2 is complete.

**Proof of Theorem 3.1**: For deducing the inequality (13) of Theorem 3.1, it suffices to show that for fixed \( \tau \), the function

\[
t \mapsto g_n(t) = e^{-2t\lambda_n^{\alpha, \beta}} \Omega_n(\tau - t)
\]

is positive on the interval \([0, \tau]\), for every integer \( n \geq 1 \) (hence \( \Omega_n \) is positive). The proof is also done by induction on \( n \). For \( n = 0 \), the property is obvious (since \( g_0(t) = 1 \)). Assume that the property is true for an integer \( n \). Then since,

\[
g'_{n+1}(t) = -2\lambda_n^{\alpha, \beta} e^{2t\lambda_n^{\alpha + \nu^+, \beta + \nu^+}} g_n(t) \leq 0,
\]

we deduce

\[
g_{n+1}(t) \geq g_{n+1}(\tau) = e^{-2\tau\lambda_n^{\alpha, \beta}} \Omega_n(0) = e^{-2\tau\lambda_n^{\alpha, \beta}} \geq 0.
\]

The proof of Theorem 3.1 is entirely completed.

We close this paper with the following concluding remarks:

**Remarque 3.3**

1. In the case \( n = 1 \), the inequality on the right side of (13) is the Poincaré inequality (12).

2. We keep entirely the notation introduced in Theorem 3.1 and Proposition 3.2. Taking account of the alternate sign in the identity (14), one may immediately deduces that, for all \( \tau \geq 0 \) and for all smooth functions \( f \) on \( I \),

\[
\langle (P_t^{\alpha, \beta} f)^2 \rangle_{\alpha, \beta} = \sum_{k=0}^{\infty} \frac{(-1)^k \Omega_k(\tau)}{\pi_k} \langle (1 - x^2)^k f(k)^2 \rangle_{\alpha, \beta}
\]

if and only if

\[
\lim_{k \to \infty} \frac{\Omega_k(\tau)}{\pi_k} \langle (1 - x^2)^k f(k)^2 \rangle_{\alpha, \beta} = 0.
\]
Références


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