A brief glimpse of extended irreversible thermodynamics theory and some of its applications

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In the last decade, it has been shown that the so-called extended irreversible thermodynamics (EIT) has a wider range of applications and a deep insight on open physical problems than classical irreversible thermodynamics in the research on processes out of equilibrium. The purpose of the present paper is to exhibit the basis and some current applications of EIT, as for instance, the generalized absolute temperature, the behavior of entropy in hyperbolic heat conduction, the flux saturation behavior and an approximate value of ballistic phase velocity.

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I. INTRODUCTION

The classical description of irreversible thermodynamics has been undoubtedly useful and has led to great progress in sciences. However, it has some drawbacks. Indeed, when the thermodynamic forces are important and at the regime of high frequencies fast phenomena appear which can not be described by the classical theory of irreversible processes [1,2]. In contrast, the Extended Irreversible Thermodynamics (EIT) theory seems to be an appropriate tool to investigate this fast phenomena domain [3,4] and to deal with the paradox of classical description concerning the infinite speed of propagation of thermal and viscous signals. Then, we report here the basis of EIT in order to make evident its usefulness. This theory, motivated by some relaxation transport equations, includes in its general version memory, non-linear and non-local effects and considers the dissipative fluxes as independent supplementary variables. Recently, many authors are interested to study how EIT could give some responses to open contemporary physical questions.

The plan of the present paper is as follows: In the second part, we deal with the classical formalism of irreversible thermodynamics recalling its basic hypothesis and we stress how much needed is for EIT. In the third part, we present the EIT theory in its simply form. In the fourth section, we discuss some applications of EIT which we consider to show the deep meaning of the theory, as for instance, the main questions arising in connection with non-equilibrium temperature are of conceptual nature and concern physical meaning of this basic quantity [5]. In the case of hyperbolic heat conduction in an isolated system [6], the local-equilibrium entropy may become negative, in contrast with the classical hypothesis. However, EIT clarified, in a simple manner, what happens at this level. Furthermore, the saturation behavior of dissipative fluxes may be treated in a hierarchy of higher moments if one generalizes the hydrodynamic transport equations by including non-local and the external field effects through the high-order moments of the velocity distribution function. However, this procedure leads to a hierarchy of variable equations which should be closed. Thus, the problem consists in finding an expression for the $(n+1)^{th}$ moment and the production terms in the corresponding evolution equation as suitable function of the first $n^{th}$ moment [7-10]. By using a continued fraction expansion, we can derive an expression of effective transport coefficients yielding to flux limiters [11] and calculating an approximate ballistic velocity of thermal pulse [12].

II. CLASSICAL FORMULATION OF NONEQUILIBRIUM THERMODYNAMICS

Classical thermodynamics was developed in nineteenth century by many well-known scientists (as Carnot, Mayer, Joule, Clausius, Lord Kelvin, Maxwell, Boltzmann, Gibbs, Planck, Duhem, etc.). As this theory concerns essentially systems at equilibrium, we shall refer to it as equilibrium thermodynamics. It is based on two main principles (first and second law) which were summarized by Clausius by means of the next statements: (1) The energy of the universe is a constant. (2) The entropy of the universe never decreases.

Afterwards, it was felt necessary to go beyond the equilibrium approach. Thus in 1931 Onsager published two pioneering papers in non-equilibrium thermodynamics [1], which were followed by other fundamental contributions by several authors as Eckart, Meixner and Prigogine [2,13,14]. The formalism proposed by them is usually referred to as Classical Irreversible Thermodynamics (CIT) and has been widely applied in Physics, Biophysics, Chemistry and Engineering science.

A. General balance laws of continuum theory

We suppose that the continuum hypothesis holds. Then, for any extensive quantity $B$, one has the following local form of the balance equation

$$\frac{\partial (\rho b)}{\partial t} = -\nabla \cdot J^b + \sigma^b$$

(1)

with $\rho$ is the total mass density, $b$ the specific value of $B$, $J^b$ represents the quantity flowing per unit area and unit time.
through the boundary and \( \sigma^t \) is the rate of production (> 0) or destruction (< 0) per unit volume of quantity \( B \).

B. The local equilibrium hypothesis

In the classical description, the states of a one component fluid out of equilibrium, are determined by a kinematic variable, the velocity field \( \mathbf{v} \) and two thermodynamics variables, for instance, the specific volume \( v \) and the specific internal energy per unit mass \( u \) [15]. The fundamental hypothesis underlying CIT is that of local equilibrium. It postulates that the local and instantaneous relations amongst the thermal and mechanical properties of a physical system are the same as for a uniform system at equilibrium. It is assumed that the system under study can be mentally split into a series of cells sufficiently large to allow them to be treated as macroscopic thermodynamic subsystems, but sufficiently small that equilibrium is very close to be realized in each cell [3]. Thus, the local equilibrium hypothesis implies that the fundamental thermodynamic equation of the system (for instance, the specific entropy per unit mass \( s \) as function of \( u \) and \( v \)) depend locally on the same variables as in equilibrium. No other variables is needed in such an equation which, furthermore, contains all of the thermodynamic information about the system. Then, the fundamental equation is

\[
s = s(u, v).
\]

C. The balance equation of mass, momentum and energy

The evolution of the system is governed by the well known balance equation of mass, momentum and energy.

As in its general form (1), one can write the balance equations in Euler representation [16,17].

\[
\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \quad (2)
\]

\[
\frac{\partial (\rho \mathbf{v})}{\partial t} = -\nabla \cdot (\mathbf{P} + \rho \mathbf{v} \mathbf{v}) + \rho \mathbf{F} \quad (3)
\]

\[
\frac{\partial (\rho u)}{\partial t} = -\nabla \cdot (q + \rho u \mathbf{v}) - \mathbf{P} : \nabla \mathbf{v} \quad (4)
\]

where \( \rho \) is the mass density, \( \mathbf{P} \) is the pressure tensor, \( \nabla \mathbf{v} \) is the velocity gradient tensor, \( \mathbf{F} \) represents interaction exerted by the external world on the inner part of the system and \( q \) describes the transport of internal energy due to conduction [3]. Expressions (2-4) are useful because they allow one to identify the various fluxes and supply terms corresponding respectively to the mass, momentum and energy, as shown in the following table,

<table>
<thead>
<tr>
<th>Quantity, ( b )</th>
<th>Flux, ( J^b )</th>
<th>Source, ( \sigma^b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass</td>
<td>( \rho \mathbf{v} )</td>
<td>0</td>
</tr>
<tr>
<td>Momentum</td>
<td>( \mathbf{P} + \rho \mathbf{v} \mathbf{v} )</td>
<td>( \rho \mathbf{F} )</td>
</tr>
<tr>
<td>Internal energy</td>
<td>( q + \rho u \mathbf{v} )</td>
<td>-( \mathbf{P} : \nabla \mathbf{v} )</td>
</tr>
</tbody>
</table>

Clearly the mass is conserved. If no external force is acting on the system, momentum is also conserved. Addition of (3), after scalar multiplication by \( \mathbf{v} \), and (4) yields the balance equation of total energy,

\[
\frac{\partial}{\partial t} [\rho (u + \frac{1}{2} \mathbf{v} \cdot \mathbf{v})] = -\nabla \cdot [q + \rho (u + \frac{1}{2} \mathbf{v} \cdot \mathbf{v}) \mathbf{v} + \mathbf{P} \mathbf{v}] + \rho \mathbf{F} \cdot \mathbf{v} \quad (5)
\]

from which it follows that in the absence of external force the total energy is conserved.

D. The law of balance entropy

In analogy with equilibrium thermodynamics, it is assumed that outside equilibrium there exists an extensive quantity \( S \), called entropy, which is a sole function of the state variables. In general, the rate of change of \( S \) can be written as the sum of two terms

\[
\frac{dS}{dt} = d^S S + d^I S \quad (6)
\]

\( d^S S/dt \) is the rate of entropy exchanged with the surroundings, which may be invariably zero, positive or negative. In contrast, the rate of entropy \( d^I S/dt \) derives from processes occurring inside the system and it is a non negative quantity according to the second law of thermodynamics,

\[
\frac{d^I S}{dt} \geq 0 \quad (7)
\]

Then, one has \( d^S S/dt = 0 \) at equilibrium or for reversible transformations and \( d^I S/dt > 0 \) for irreversible processes. Since \( S \) is an extensive quantity, its specific value \( s \) will obey the general form of balance equation (1) expressed as

\[
\frac{\partial (\rho s)}{\partial t} = -\nabla \cdot (\mathbf{J}^s + \rho s \mathbf{v}) + \sigma^s \quad (8)
\]

with

\[
\sigma^s \geq 0 \quad (9)
\]

In either case, inequality (9) goes beyond the usual formulation of the second law in equilibrium thermodynamics where only the global increase of entropy between two equilibrium states in the system is considered. Here it is assumed that the statement (9) holds at any position in space and any instant of time, for whatever the evolution of the system. One of the main objectives of non-equilibrium thermodynamics is to express \( \sigma \) as function of the quantities characterizing the irreversible processes. This is important because it displays the sources of irreversibility occurring in a process.

E. Gibbs equation and constitutive equation

The classical description of fluid mechanics is not self contained, since the evolution equations (2-4) involve new unknowns as \( q, p^s \) and \( \mathbf{P} \cdot \mathbf{v} \). To overcome this difficulty, the set of equations is closed by means of the so-called equations of state and constitutive equations, the former
expressions \( p^v \) and the latter \( q \) and \( P^v \) in terms of basic variables and their spatial gradients.

According to the hypothesis of local equilibrium, the Gibbs equation formulate in thermostat writes as

\[
ds = T^{-1} d \mathbf{u} + T^{-1} p \, d \mathbf{v}
\]

(10)

in which \( T \) is the absolute temperature. A direct differentiation of \( s \) with respect to \( u \) and \( v \)

\[
ds = \left( \frac{\partial s}{\partial u} \right)_v \, du + \left( \frac{\partial s}{\partial v} \right)_u \, dv
\]

(11)

which yields the equation of state for \( T \) and \( p \), namely

\[
T^{-1}(u,v) = \left( \frac{\partial s}{\partial u} \right)_v
\]

(12a)

\[
T^{-1} p(u,v) = \left( \frac{\partial s}{\partial v} \right)_u
\]

(12b)

In view of (10), the time derivative of the entropy is given in Lagrange's representation by

\[
\rho \dot{s} = T^{-1} \rho \mathbf{u} + T^{-1} \rho \mathbf{v} \cdot \nabla \mathbf{v}
\]

(13)

Introduction of (3-4) into (13) leads to

\[
\rho \dot{s} = -\nabla \cdot (T^{-1} \mathbf{q}) - T^{-2} \mathbf{q} \cdot \nabla T
\]

\[
- T^{-1} p^v \nabla \cdot \mathbf{v} - T^{-1} P^v \cdot \mathbf{v}
\]

(14)

The most general form of the entropy balance equation is written

\[
\rho \dot{s} + \nabla \cdot \mathbf{J}^s = \sigma^t
\]

(15)

After comparing (14) and (15) one is led to

\[
\mathbf{J}^s = T^{-1} \mathbf{q}
\]

(16)

and

\[
\sigma^t = -T^{-2} \mathbf{q} \cdot \nabla T - T^{-1} p^v \nabla \cdot \mathbf{v}
\]

\[
- T^{-1} P^v \cdot \mathbf{v}
\]

(17)

It is usual to call \( q \), \( p^v \) and \( P^v \) dissipative fluxes, whereas \( T^{-2} \nabla T \), \( T^{-1} \nabla \cdot \mathbf{v} \) and \( T^{-1} \mathbf{v} \) are the thermodynamic forces. It is observed that the expression (17) is a bilinear form in thermodynamic fluxes and forces. The local version of the second law states that entropy production is positive definite. Then, the most direct way to relate the fluxes to the forces in a manner that ensures the positive character of \( \sigma^t \) is to assume linear relations between them, i.e.

\[
q = -\mu_1 T^{-2} \nabla T
\]

(18)

\[
p^v = -\mu_0 T^{-1} \nabla \cdot \mathbf{v}
\]

(19)

\[
P^v = -\mu_2 T^{-1} \mathbf{v}
\]

(20)

These relations are directly comparable to the Fourier and Stockes-Newton equations, so that the coefficients \( \mu_0 \) and \( \mu_2 \) may be related to the usual thermal conductivity \( \lambda \), bulk viscosity \( \zeta \) and shear viscosity \( \eta \), respectively, by

\[
\mu_0 = \frac{\lambda}{T}, \quad \mu_2 = \frac{\zeta}{T}
\]

The restrictions \( \sigma^t \geq 0 \) imposes, as easily seen by introducing (18-20) into (17), that \( \mu_0 > 0 \), \( \mu_2 > 0 \) and \( \mu_2 \geq 0 \), expressing the well known positive semi definite character of the transport coefficients.

F. Limitations of the classical description

In this paragraph, we talk about some inconvenient of the classical theory. Indeed, the classical description has led to an impressive production of scientific work [14,16-20]. However, it has some drawbacks from fundamental and applied points of view:

i- It is based on the a priori local-equilibrium hypothesis, which is too restrictive, for a wide class of phenomena. It is conceivable, indeed, that other variables, not found in equilibrium, may influence the thermodynamic equations in non-equilibrium situation.

ii- The constitutive equations are assumed to be linear. In several cases, this is in contradiction to experimental observation, the more outstanding ones being those corresponding to non-Newtonian fluids or to some electrical devices.

iii- The constitutive equations are not satisfactory at high frequencies and short wavelengths, as manifested in experiments on ultrasound propagation in rarefied gases, on neutron scattering in liquids and on phonon propagation in solids at low temperatures. Moreover, the classical constitutive equations lead, when introduced into the balance equations, to parabolic equations implying infinite speed of propagation of thermal and viscous signals.

iv- Very intensive work from the microscopic point of view has led to constitutive equations which are far more general than those of the classical theory. Such equations contain either relaxational and/or non-linear terms which are absent in the classical formalism. Further, the microscopic description has lacked a macroscopic counterpart, and the thermodynamic formalism corresponding to generalize constitutive equations is needed. Therefore, this is the objective of Extended Irreversible Thermodynamics which is able to deal with the above mentioned limitations.

III. EXTENDED IRREVERSIBLE THERMODYNAMICS

Extended irreversible Thermodynamics (EIT) is a theory which goes beyond the classical description. This theory enlarges the space of basic independent variables through the introduction of non-equilibrium variables, such as the dissipative fluxes appearing in the balance equations.
(24). Of course, it is expected that the results of classical theory will be recovered as a limiting case. The next step in the formulation of the theory is to find evolution equations for these extra variables where the restrictions on the form of evolution equations will be imposed by the second law of thermodynamics.

A. Generalized Gibbs equation

Our starting hypothesis is to consider the fluxes as independent variables, so that it is logical to include them among the set of variables on which the entropy depends.

Thus, we assume that entropy of the system depends locally not only on the classical variables (internal energy $u$ and specific volume $v$ as in the previous section) but also on the dissipative fluxes $q$, $p^v$ and $P^v$. We will show in section V an example in which a high number of fluxes is taken into account as independent parameters in order to clarify the saturation behavior phenomena.

The hypothesis of a generalized macroscopic entropy depending on the fluxes was advanced in a restricted way in 1953 [1]. During the 1960s a more direct formalism applied to fluids was developed [21]. In the 1970s, new reformulations were rediscovered independently by several authors and have inspired much research which has been reviewed [3]. We assume then the generalized entropy to be $s = s (u, v, q, p^v, P^v)$. In its differential form, the specific entropy is written

$$ds = \left(\frac{\partial s}{\partial u}\right) du + \left(\frac{\partial s}{\partial v}\right) dv + \left(\frac{\partial s}{\partial q}\right) dq + \left(\frac{\partial s}{\partial p^v}\right) dp^v + \left(\frac{\partial s}{\partial P^v}\right) dP^v$$

(21)

In analogy with the classical theory, we define a non-equilibrium absolute temperature $\theta$ and a non-equilibrium thermodynamic pressure $\pi$:

$$\theta^{-1}(u, v, q, p^v, P^v) = \left(\frac{\partial s}{\partial u}\right)_{v, q, p^v, P^v}$$

$$\theta^{-1}(u, v, q, p^v, P^v) = \left(\frac{\partial s}{\partial v}\right)_{u, q, p^v, P^v}$$

(22a, 22b)

The quantities $\pi$ and $\theta$ can be expanded around their local-equilibrium counterpart $T$ and $p$ according to

$$\theta^{-1} = T^{-1}(u, v) + O(q, q, p^v, p^v, P^v, P^v)$$

$$\theta^{-1} = T^{-1}(u, v) + O(q, q, p^v, p^v, P^v, P^v)$$

(23a, 23b)

when $O$ refer to correction terms which are of second order in the fluxes. They are usually negligible and will therefore be omitted. The remaining partial derivatives in (21) are denoted

$$\frac{\partial s}{\partial q} = \theta^{-1} v_1 (u, v, q, p^v, P^v)$$

$$\frac{\partial s}{\partial p^v} = \theta^{-1} v_0 (u, v, q, p^v, P^v)$$

$$\frac{\partial s}{\partial P^v} = \theta^{-1} v_0 (u, v, q, p^v, P^v)$$

(24a, 24b, 24c)

with $\alpha_1$, $\alpha_0$, and $\alpha_2$ are vector, scalar and tensor respectively.

For isotropic systems, $T$, $p$, and $\alpha_i$, are functions of $u$, $v$ and the algebraic invariant of $q$ and $p^v$. For simplicity, we assume that the coefficients $\alpha_1$, $\alpha_0$, and $\alpha_2$ are linear in the fluxes, so that

$$\alpha_1 = \alpha_1^0 v, \quad \alpha_0 = \alpha_0^0 v, \quad \alpha_2 = \alpha_2^0 v$$

(25)

where $\alpha_1^0$, $\alpha_0^0$, and $\alpha_2^0$ are scalar functions of $u$ and $v$.

Introducing (22-25) into (21) yields the generalized Gibbs equation:

$$ds = T^{-1} du + T^{-1} p dv + \theta^{-1} \alpha_1 v_1 q, dq$$

$$-T^{-1} \alpha_0 v_0 p^v dp^v - T^{-1} \alpha_2 v_2 P^v dP^v$$

(26)

Of course, this relation is useless unless the coefficients $\alpha_1$, $\alpha_0$, and $\alpha_2$ are identified in terms of physical parameters, this is done in the next paragraph.

From (26) and the balance equation of energy and mass, one obtains for $s$, the material time derivative of $s$,

$$\rho s = -T^{-1} V_q q - T^{-1} p^v V_u - T^{-1} P^v V_{P^v}$$

$$-T^{-1} \alpha_1^0 q \dot{q} - T^{-1} \alpha_0^0 p^v \dot{p^v}$$

$$-T^{-1} \alpha_2^0 P^v \dot{P^v}$$

(27)

This equation can be cast in the general form of a balance equation,

$$\rho \dot{s} + V J^s = \sigma^s$$

(28)

on the condition that one can identify the expression for the entropy flux $J^s$ and the entropy production $\sigma^s$. 
B. Generalized entropy flux and entropy production

For isotropic system, the most general vector depending on the variables \( u, v, q, p^\nu \) and \( P^\nu \), is up to second-order terms in the fluxes

\[
J^\nu = \beta q + \beta' P^\nu q + \beta'' P^\nu \cdot q
\]  
(29)

where the coefficients \( \beta, \beta' \) and \( \beta'' \) are functions of \( u \) and \( v \). The coefficient \( \beta \) must be identified as \( T^{-1} \) to recover the results of the classical theory of irreversible processes.

The entropy production is derived from the general form of the balance equation (27). Introduction (27) and (29) into (28) leads to the following expression for the entropy production,

\[
\sigma^\nu = q \cdot [\nabla T^{-1} + \beta' \nabla p^\nu + \beta'' \nabla P^\nu - T^{-1} \alpha_{10} q] + p^\nu [-T^{-1} \nabla q - T^{-1} \alpha_{00} p^\nu + \nabla (\beta' q)] + P^\nu [-T^{-1} \nabla T^{-1} \alpha_{21} (P^\nu)'] + [\nabla (\beta'' q)]
\]  
(30)

One observes that (30) has the structure of bilinear form

\[
\sigma^\nu = q \cdot X_1 + p^\nu X_0 + P^\nu \cdot X_2
\]  
(31)

consisting of a sum of products of fluxes \( q, p^\nu \) and \( P^\nu \) and their conjugate generalized forces \( X_1, X_0 \) and \( X_2 \). The latter follows from direct comparison of (II.11) with (30). They are similar to the expression obtained in CIT but contain additional terms depending on the time and space derivatives of the fluxes.

In order to obtain evolution equations for the fluxes compatible with the positiveness of \( \sigma^\nu \), we express the forces \( X_1, X_0 \) and \( X_2 \) as functions of the fluxes. We shall assume linear relation between fluxes and forces to avoid complications in calculations and propose the EIT theory on its simple form. Thus, we have

\[
X_1 = \mu_1 q, \quad X_0 = \mu_0 p^\nu, \quad X_2 = \mu_2 P^\nu
\]  
(32)

with \( \mu_i = \mu_i (u, v), \) \((i=1, 2, \ldots)\). Introduction (32) into (31) yields

\[
\sigma^\nu = \mu_1 q + \mu_0 P^\nu + \mu_2 P^\nu \cdot P^\nu
\]  
(33)

Then, the restrictions \( \sigma^\nu \geq 0 \) implies that \( \mu_i \geq 0 \), \( \mu_0 \geq 0 \) and \( \mu_2 \geq 0 \). These restrictions yield the simplest form of the evolution equations compatible with the second law of thermodynamics.

C. Evolution Equation of the Fluxes

Identifying the forces as the conjugate terms of the fluxes (30) and substituting these expressions in (32), one obtains in the linear approximation (products like \( q \cdot V^\nu \) and \( q \cdot V^\nu \) are omitted) the following set of evolution equations

\[
\nabla T^{-1} - T^{-1} \alpha_{10} q = \mu_1 q - \beta' \nabla p^\nu - \beta'' \nabla P^\nu
\]  
(34)

\[
- T^{-1} \nabla q - T^{-1} \alpha_{00} p^\nu = \mu_0 p^\nu - \beta' \nabla q
\]  
(35)

\[
- T^{-1} \nabla T^{-1} \alpha_{21} (P^\nu)' = \mu_2 P^\nu - \beta'' (\nabla q)
\]  
(36)

One of the feature issued from this thermodynamic formalism, which is confirmed by kinetic theory, concerns a generalized of Onsager's relations by mean of equations (34-36). This generalization affects the cross term relating \( q \) with \( \nabla P^\nu \) and \( \nabla \cdot (q \cdot \nabla) \) on one side, \( q \) with \( \nabla p^\nu \) and \( p^\nu \) with \( \nabla q \) on the other. The equality such coefficients, which has been obtained here from purely thermodynamic arguments, is supported by kinetic theory [22-26].

D. Identification of the coefficients

We have introduced several coefficients, which must be identified on physical grounds. For this we assume two kind of situations,

1) Stationary and homogeneous fluxes

Equations (34-36) reduce then to

\[
\nabla T^{-1} = \mu_1 q
\]  
(37a)

\[
- T^{-1} \nabla q = \mu_0 p^\nu
\]  
(37b)

\[
- T^{-1} \nabla = \mu_2 P^\nu
\]  
(37c)

comparison with the well known Fourier and Newton-Stokes laws, i.e. respectively

\[
q = -\lambda \nabla T, \quad p^\nu = -\xi \nabla q \quad \text{and} \quad P^\nu = -2\eta \nabla q
\]

leads to the identifications

\[
\mu_1 = (\lambda T^2)^{-1}, \quad \mu_0 = (\xi T)^{-1}, \quad \mu_2 = (2\eta T)^{-1}
\]  
(38)

with \( \lambda, \xi \) and \( \eta \) the thermal conductivity, bulk viscosity and shear viscosity respectively.

2) Non stationary and homogeneous fluxes

In this situation (34-36) reduce to

\[
\nabla T^{-1} - T^{-1} \alpha_{10} q = (\lambda T^2)^{-1} q
\]  
(39)

\[
- T^{-1} \nabla q - T^{-1} \alpha_{00} p^\nu = (\xi T)^{-1} p^\nu
\]  
(40)

\[
- T^{-1} \nabla T^{-1} \alpha_{21} (P^\nu)' = (2\eta T)^{-1} P^\nu
\]  
(41)

This set of equations can be identified with the so-called Maxwell-Cattaneo laws,
\[ \tau_1 q + q = -\lambda \nabla T \]  \hspace{1cm} (42)

\[ \tau_0 p^v + p^v = -\xi \nabla u \]  \hspace{1cm} (43)

\[ \tau_2 (\dot{p}^v) + p^v = -2 \eta \nabla \]  \hspace{1cm} (44)

where \( \tau_1, \tau_2 \) and \( \tau_3 \) are the relaxation times of respective fluxes. We are then led to the identifications

\[ \alpha_{10} = \tau_1 (\lambda T)^{-1}, \alpha_{00} = \tau_0 \xi^{-1}, \alpha_{21} = \tau_2 \left( 2 \eta \right)^{-1} \]  \hspace{1cm} (45)

Therefore, the evolution equations (34-36) take the following form,

\[ \tau_1 \dot{q} = - (q + \lambda \nabla T) + \beta' \lambda T^2 \nabla p^v + \beta' \lambda T^2 \nabla \dot{p}^v \]  \hspace{1cm} (46)

\[ \tau_0 \dot{p}^v = - (p^v + \xi \nabla u) + \beta' \xi T \nabla q \]  \hspace{1cm} (47)

\[ \tau_2 (\dot{p}^v) = - (2 \eta \nabla q) + \beta' \eta T (\nabla q) \]  \hspace{1cm} (48)

Note that when relaxation times diverge but the ratio of relaxation time/transport coefficient (\( \lambda, \xi, \eta ... \)) remains finite or in the high-frequency regime (\( \tau_0 >> 1 \)), see section 1, the fokker-planck equation becomes reversible because the term in time derivative of the fluxes is much more important than the fluxes themselves. In these particular circumstances, there is no dissipation associated to the fluxes (e.g. electric current in superconductors). Then by substituting (45) into (26),

\[ ds = \theta^{-1} du + \theta^{-1} \pi dv - \left( \frac{\nu T_1}{T^2} \right) q . dq \]

\[ - \left( \frac{\nu T_0}{\xi T} \right) p^v dp^v + \left( \frac{\nu T_2}{2 \xi T} \right) \nabla \dot{p}^v \cdot d \nabla \]  \hspace{1cm} (49)

where \( \theta \) is the generalized absolute temperature and the thermodynamic pressure \( \pi \) instead of their respective local-equilibrium approximation \( T \) and \( p \). It is seen that the presence of nonvanishing relaxation times implies a modification of the classical Gibbs equation, which is recovered when \( \tau_i \to 0 \).

V. NON-EQUILIBRIUM ABSOLUTE TEMPERATURE

We consider a rigid solid which according to the hypothesis of EIT, is locally characterized by the specific internal energy \( u \) per unit mass and the heat flux vector \( q \). Then, the generalized Gibbs equation (49) can be written as

\[ ds = \theta^{-1} du - \left( \frac{\nu T_1}{T^2} \right) q . dq \]  \hspace{1cm} (50)

\[ [T^{-1}] = [T_{syst}]^{-1} - [T_{therm}]^{-1} = \frac{1}{2} \left[ \left( \frac{\nu T_1}{\lambda T^2} \right) q . q \right]_{syst} - \left( \frac{\nu T_1}{\lambda T^2} \right) q . q_{therm} \]  \hspace{1cm} (54)

The above analysis shows that in non-equilibrium states, the generalized absolute temperature \( \theta \) has a deeper meaning than the local equilibrium temperature \( T \). We do not forget to note now that the presence of a generalized temperature is

Up to now we have neglected the terms of order \( q . q \) (see (24) in the expression of non-equilibrium temperature). Here, we examine the consequences of including them. The generalized Gibbs equations (50) has as the integrability condition

\[ \frac{\partial \theta^{-1}}{\partial q} = - \left( \frac{\nu T_1}{\lambda T^2} \right) q \]  \hspace{1cm} (51)

Integration of this equation leads for absolute temperature to

\[ \theta^{-1} (u, v) = T^{-1} (u) - \frac{1}{2} \left( \frac{\nu T_1}{\lambda T^2} \right) q . q \]  \hspace{1cm} (52)

with \( T(u) \) the local equilibrium temperature, which depends only on the internal energy.

The main question arising in connection with (52) is of a conceptual nature and concerns the physical meaning of the temperature. In this connection, let us study the behavior of \( \theta \) across an ideal wall (a wall where no entropy production is produced) between a thermometer and a system [27].

Consider a thermometer immersed in a system out of equilibrium in which there is a heat flux. Assume that the intervening wall between the thermometer and the system is ideal. The energy and the entropy balance equations across such a discontinuity take the general form,

\[ \rho \dot{u}_{wall} + [q . n] = 0 \]

\[ \rho \dot{s}_{wall} + [S . n] = \sigma_{wall} \]  \hspace{1cm} (53)

where the square bracket stands for the discontinuity of the corresponding quantity across the wall and \( n \) is the unit normal.

Because of the ideal character of the wall \( \sigma_{wall} = 0 \). Then, in steady state (53) may be written simply as \([q . n] = 0 \) and \([\theta . q . n] = 0 \). As a consequence is turns out that the generalized temperature \( \theta \) is continuous across the ideal wall

\[ \theta^{-1} = \theta^{-1}_{system} - \theta^{-1}_{thermometer} = 0 \]

Therefore, the thermometer and the system have the same non-equilibrium temperature \( \theta \), but their respective local equilibrium absolute temperature are different. According to (52), the discontinuity in the local equilibrium absolute temperature is

\[ \theta^{-1} = \theta^{-1}_{system} - \theta^{-1}_{thermometer} = 0 \]
not restricted to extended irreversible thermodynamics but there are several contributions on this topic [3,7,28-30].

V. BEHAVIOR OF ENTROPY IN HYPERBOLIC HEAT CONDUCTION

In this section, we present an important contribution of EIT to clarify the behavior of entropy in hyperbolic heat conduction of an isolated system. Indeed, relaxational equations for the heat flux $q$, of which the simplest one would be the Maxwell-Cattaneo equation [6].

$$\tau \dot{q} + q = -\lambda \nabla T$$  \hspace{1cm} (55)

lead to hyperbolic equations for the temperature $T$ as, for instance, in the case of constant $\lambda$ (thermal conductivity) and $\tau$ (relaxation time)

$$\frac{\partial^2 T}{\partial t^2} + \frac{\partial T}{\partial t} = \chi \nabla^2 T$$  \hspace{1cm} (56)

with $\chi = \lambda / (\rho c)$ the thermal diffusivity, $\rho$ being the mass density and $c$ the heat capacity per unit mass. The behavior of the generalized entropy along the evolution of an isolated rigid system towards thermal equilibrium is compared with the local-equilibrium entropy [6]. In contrast with the local-equilibrium entropy which becomes negative at several stages of evolution, the generalized entropy of extended irreversible thermodynamics exhibits a monotone increase with some stationary points, corresponding to the minima of the local-equilibrium entropy. So, it is clear that the classical entropy of an isolated system does not increase monotonically when the approach to equilibrium is described by hyperbolic equations for temperature evolution. However, the generalized entropy of extended irreversible thermodynamics does increase monotonically, and therefore better suited than the classical theory for the description of hyperbolic heat conduction.

VI. FLUX SATURATION PHENOMENA

$$J^{(m-p)} = J_{\sigma}^{(m-p-1)} \frac{i \dot{\lambda}_{m-p} k}{1 + i \omega \tau_{m-p} + \frac{\gamma_{m-p} \dot{\lambda}_{m-p+1} k^2}{1 + i \omega \tau_{m-p+1} + \frac{\gamma_{m-p+1} \dot{\lambda}_{m-p+2} k^2}{1 + i \omega \tau_{m-p+2} + \cdots}} \cdots \frac{1 + i \omega \tau_{m-1} + \gamma_{m-1} \dot{\lambda}_m k^2}{1 + i \omega \tau_m}}$$  \hspace{1cm} (59)

The above continued-fraction expansion leads to the flux limiters for system under the influence of an external field by using the essential changes of parameters of system. For example, if we consider the system under the influence of electric fields $E$ or under a temperature gradient $\nabla T$, we obtain in the steady state flux-limited expression for the electric current $J$ and the heat flux $Q$ respectively.

Using a continued-fraction expansion in the framework of EIT, one will show the flux saturation behavior by considering a hierarchy of evolution equations for higher-order moments.

The flux saturation behaviour, which is well known, for instance, in radiation hydrodynamics [8,9,31], in plasma physics [32,33] and in electronic [3,10], is an important non-linear feature which should be taken into account in the formulation of transport laws valid at high values of thermodynamic forces. For instance, from information theory and fluctuation-dissipative theory, the identification of the meaning of the Lagrange multiplier corresponding to the heat flux yields an effective thermal conductivity [34,35], which leads itself to a heat flux limiter, which is closed to other flux limiters found in the literature on heat transport, plasma physics, radiation hydrodynamics, etc.

In the framework of the EIT, the generalized Gibbs equation is written [36,37],

$$ds = 0^{-1} d\mu - 0^{-1} \mu d\eta = \sum_{n=1}^{\infty} \alpha_n J^{(n)} dJ^{(n)}$$  \hspace{1cm} (57)

where $u$ is internal energy, $n$ the particle density, $\theta$ non-equilibrium temperature, $\mu$ non-equilibrium chemical potential, $\alpha_n$ coefficient depending on $u$ and $n$, and, $J^{(n)}$ is the $n^{th}$ flux. Then, the simplest form of the flux evolution equation, compatible with the second law of thermodynamics, can be generalized in the compact form as

$$\tau_m \frac{\partial J^{(m)}}{\partial t} + J^{(m)} = \lambda_m \nabla J^{(m-1)} + \gamma_m \nabla J^{(m+1)}$$  \hspace{1cm} (58)

with $\tau_m$ is the relaxation time, $\lambda_m$ and $\gamma_m$ are transport coefficients ($m=1,2,\ldots$).

If we write the last equation (58) in Fourier-Laplace space ($\alpha k$) and truncate the hierarchy at $(m+1)^{th}$ order, one could obtain the next continued-fraction expansion (59),

$$J = \frac{\sigma_{\alpha} E}{1 + \frac{1}{4} - \frac{1}{4} a \sigma_{\alpha} E^2}$$  \hspace{1cm} (60)

$$Q = \frac{-\lambda \nabla T}{1 + \frac{1}{4} + \frac{l^2}{4} (\nabla ln(T))}$$  \hspace{1cm} (61)
where \( \sigma_0 \) is the electric conductivity, \( a \) is negative constant, \( \lambda \) is the thermal conductivity and \( l \) the mean free path.

VII. TRANSITION FROM DIFFUSION TO BALLISTIC TRANSPORT

Generally, there are three mechanisms of particles transport. One should distinguish among ballistic, second sound and diffusion transports. These mechanisms depend on temperature and purity of crystals. At low frequencies, one should observe diffusion transport. At intermediate frequencies, we have second sound transport and at high frequencies the ballistic transport exists [38]. A ballistic velocity may be derived by using the continued-fraction techniques [39] described in section V. By truncating the continued-fraction at the second order, it is shown the next dispersion relation,

\[
i\omega = \frac{-\xi_2 k^2}{i\omega \tau_1 + \frac{2l_2^2 k^2}{4l_2^2 - \omega^2 \tau_2^2}}
\]

(62)

with

\[
\xi_2 = \frac{\lambda_c}{\rho c},
\]

\[
l_2^2 = \lambda_c \gamma_n
\]

and

\[
\left( \frac{l}{\tau} \right)^2 = \frac{(n + 1)^2}{4(n + 1)^2 - 1} c_o^2
\]

where \( \rho \) and \( c_o \) denote respectively density of mass and heat capacity.

If we assume \( x = \frac{k}{\omega} \) then the equation (62) implies the following expression depending on \( \tau_1 / \tau_2 \)

\[
x^4 + a(\tau_1 / \tau_2) x^2 + b(\tau_1 / \tau_2) = 0
\]

(63)

In fact, the propagation speed should be lower than or equal to the limit velocity \( c_o \) (Debye's velocity or light velocity according to the system considered). Indeed, the propagation speed depends on the time relaxation of the processes occurring within systems [12,40,41]. At high frequencies, we have just improved here (solution of the equation (VI.2)) the value of phase velocity as 1.012 \( c_o \), which is a better approximate value than 1.026 \( c_o \) calculated in the case where it is assumed equality among all relaxational times and all correlation lengths.

VIII. CONCLUSIONS

Although the theory is much wider than the content of this short presentation, we have tried to show a glimpse of some aspects of EIT. Let us summarize the main ideas: (1) the fluxes are considered as additional independent variables; (2) the entropy and the entropy flux are generalized with respect to their local-equilibrium expressions; (3) the evolution equations for the fluxes generalize the classical transport laws by including memory, non-local and non-linear effects.

Its microscopic foundations have been analyzed from kinetic theory of gases and of polymers, from information theory, from fluctuation theory, from projection-operator techniques and from computer simulations. In this connection, the microscopic justification of EIT is more demanding than that of classical theory of irreversible processes, and it encompasses a wider range of microscopic techniques; several open problems are in this context, which stimulate a closer attention on the foundations of the thermodynamics of non-equilibrium processes.

The theory has not only a fundamental interest. Its application cover a wide range of situations; as it was mentioned in the introduction. The analysis of transport in nanometric electronic devices, the shear-induced changes in the phase diagram of polymer solutions under flow, shear-induced diffusion, and nonchemical heat transport (phonon hydrodynamics, second sound, ballistic transport) are at present the closest to practical interests. Other active applications are found in viscous cosmological models and nuclear collisions, which belong to a more speculative domain.

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