Wilson’s functional equation with an endomorphism

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Abstract

In the present paper, we determine the complex-valued solutions \((f, g)\) of the functional equation

\[ f(xy) + f(\varphi(y)x) = 2f(x)g(y), \]

in the setting of groups and monoids, where \(\varphi\) is an endomorphism not necessarily involutive. We prove that their solutions can be expressed in terms of multiplicative and additive functions. Many consequences of these results are presented.

Key words: Functional equation, Wilson, endomorphism, monoid, multiplicative function.

1 Introduction

Lemma 1.1.

The functional equation

\[ g(x + y) + g(x - y) = 2g(x)g(y), \quad x, y \in \mathbb{R}, \quad (1.1) \]

is known as the d’Alembert’s functional equation. It has a long history going back to d’Alembert [3]. As the name suggests this functional equation was introduced by d’Alembert in connection with the composition of forces and plays a central role in determining the sum of two vectors in Euclidean and non-Euclidean geometries. The continuous solutions of (1.1) were determined by Cauchy in 1821 (see [1]).

In [15], Wilson dealt with a functional equation related to and generalizing d’Alembert’s functional equation (1.1). He generalized (1.1) to

\[ f(x + y) + f(x - y) = 2f(x)g(y), \quad x, y \in \mathbb{R}, \quad (1.2) \]

that contains the two unknown functions \(f\) and \(g\).
d’Alembert’s functional equation (1.1) possesses periodic and not periodic solutions. To exclude the non-periodic solutions Kannappan, in 1968, modified (1.1) to the functional equation

\[ f(x - y + z_0) + f(x + y + z_0) = 2f(x)f(y), \quad x, y \in \mathbb{R}, \]  

(1.3)

where \( z_0 \) is a non-zero real constant (we have replaced Kannappan’s notation 2A by \( z_0 \)). Kannappan proved that any solution \( f : \mathbb{R} \to \mathbb{C} \) of (1.3) has the form \( f(x) = g(x - z_0) \), where \( g : \mathbb{R} \to \mathbb{C} \) is a periodic solution of (1.1) with period \( 2z_0 \). This enabled him to find all Lebesgue measurable solutions (see [9]).

Equations (1.1), (1.2), and (1.3) have been extended to abelian groups: You just replace the domain of definition \( \mathbb{R} \) by an abelian group \( G \). The purpose of the present paper is to solve the following Wilson type functional equation

\[ f(xy) + f(\varphi(y)x) = 2f(x)f(y), \quad x, y \in S, \]  

(1.4)

where \( S \) is a semigroup and \( \varphi : S \to S \) is an endomorphism. This equation provides a common generalization of (1.1), the symmetrized multiplicative Cauchy equation

\[ g(xy) + g(yx) = 2g(x)g(y), \quad x, y \in S, \]

and the variant of d’Alembert’s functional equation

\[ g(xy) + g(\sigma(y)x) = 2g(x)g(y), \quad x, y \in S, \]

which was introduced and solved by Stetkær in [14].

The purpose of the present paper is to solve the following Wilson type functional equation

\[ f(xy) + f(\varphi(y)x) = 2f(x)f(y), \quad x, y \in M, \]  

(1.5)

where \( M \) is a possibly non-abelian group or monoid (that is, a semigroup with identity) and \( \varphi : M \to M \) is an endomorphism, for unknown functions \( f, g : M \to \mathbb{C} \). This equation, in the case where \( \varphi \) is an involutive automorphism, has been introduced and solved by Fadli et al. in [5].

By elementary methods we find all solutions of (1.5) on monoids that are generated by their squares and on groups, in terms of multiplicative and additive functions. This contrasts the solutions of the functional equation \( f(xy) + f(y^{-1}x) = 2f(x)g(y) \), where the non-abelian phenomena like 2-dimensional irreducible representations may occur (see [8]). Our results constitute a natural extension of earlier results of, e.g., [5].

As other important results in this paper, we solve the following Kannappan type functional equations

\[
\begin{align*}
 f(xyz_0) + f(\varphi(y)xz_0) &= 2f(x)g(y), \quad x, y \in G, \\
 f(xyz_0) + f(\varphi(y)xz_0) &= 2g(x)f(y), \quad x, y \in G, \\
 f(xy_0z_0) + f(\varphi(y)yz_0) &= 2f(x)f(y), \quad x, y \in G,
\end{align*}
\]

where \( G \) is a group, \( z_0 \in G \) is a fixed element, and \( \varphi : G \to G \) is an endomorphism.

Finally, we note that the sine addition law on semigroups given in [7] is a key ingredient of the proof of our main results (Theorems 3.3 and 3.4).

2 Notation and terminology

To formulate our results we recall the following notation and assumptions that will be used throughout the paper:

Let \( S \) be a semigroup, i.e., a non-empty set equipped with an associative operation. A monoid \( M \) is a semigroup with an identity element that we denote \( e \).

The map \( \sigma : S \to S \) denote an involutive automorphism. That \( \sigma \) is involutive means that \( \sigma(\sigma(x)) = x \) for all \( x \in S \). If \( (G, +) \) is an abelian group, then the inversion \( \sigma(x) := -x \) is an example of an involutive automorphism. Another example is the complex conjugation map on the multiplicative group of non-zero complex numbers.

A function \( A : S \to \mathbb{C} \) is called additive, if it satisfies \( A(xy) = A(x) + A(y) \) for all \( x, y \in S \).
A multiplicative function on $S$ is a map $\chi : S \to \mathbb{C}$ such that $\chi(xy) = \chi(x)\chi(y)$ for all $x, y \in S$. A character on a group $G$ is a homomorphism from $G$ into the multiplicative group of non-zero complex numbers. While a non-zero multiplicative function on a group can never take the value 0, it is possible for a multiplicative function on a monoid to take the value 0 on a proper, non-empty subset of $S$. If $\chi : S \to \mathbb{C}$ is multiplicative and $\chi \neq 0$, then

$$I_\chi = \{x \in S \mid \chi(x) = 0\}$$

is either empty or a proper subset of $S$. The fact that $\chi$ is multiplicative establishes that $I_\chi$ is a two-sided ideal in $S$ if not empty (for us an ideal is never the empty set). It follows also that $S \setminus I_\chi$ is a subsemigroup of $S$. These ideals play an essential role in our discussion of equation (1.5) on monoids.

If $S$ is a topological space, then we let $C(S)$ denote the algebra of continuous functions from $S$ into $\mathbb{C}$.

### 3 Main results

We first give a result for the sine addition law on semigroups.

**Lemma 3.1** ([7], Lemma 3.4). Let $S$ be a semigroup, and suppose $f, g : S \to \mathbb{C}$ satisfy the sine addition law

$$f(xy) = f(x)g(y) + f(y)g(x), \quad x, y \in S,$$

with $f \neq 0$. Then there exist multiplicative functions $\chi_1, \chi_2 : S \to \mathbb{C}$ such that

$$g = \frac{\chi_1 + \chi_2}{2}.$$

Additionally we have the following.

(i) If $\chi_1 \neq \chi_2$, then $f = c(\chi_1 - \chi_2)$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

(ii) If $\chi_1 = \chi_2 = \chi$, then $g = \chi$. If $S$ is a semigroup such that $S = \{xy \mid x, y \in S\}$ (for instance a monoid), then $\chi \neq 0$.

If $S$ is a group, then there is an additive function $A : S \to \mathbb{C}$, $A \neq 0$, such that $f = \chi A$.

If $S$ is a semigroup which is generated by its squares, then there exists an additive function $A : S \setminus I_\chi \to \mathbb{C}$ for which

$$f(x) = \begin{cases} 
\chi(x)A(x) & \text{for } x \in M \setminus I_\chi \\
0 & \text{for } x \in I_\chi
\end{cases}$$

Furthermore, if $S$ is a topological group, or if $S$ is a topological semigroup generated by its squares, and $f, g \in C(S)$, then $\chi_1, \chi_2, \chi \in C(S)$. In the group case $A \in C(S)$ and in the second case $A \in C(S \setminus I_\chi)$.

The following lemma will be used in the proof of our main results (Theorems 3.3 and 3.4).

**Lemma 3.2** ([6], Theorem 3.1). Let $S$ be a semigroup and let $\varphi : S \to S$ be an endomorphism. The solutions $g : S \to \mathbb{C}$ of the functional equation (1.4) are the following:

(i) There exists a non-zero multiplicative function $\chi : S \to \mathbb{C}$ with $\chi \circ \varphi = 0$ such that $g = \frac{1}{2}\chi$.

(ii) There exists a multiplicative function $\chi : S \to \mathbb{C}$ with $\chi = \chi \circ \varphi^2$ such that $g = \frac{1}{2}(\chi + \chi \circ \varphi)$.

Moreover, if $S$ a topological semigroup and $g \in C(S)$, then $\chi, \chi \circ \varphi \in C(S)$.

The following theorem solves the functional equation (1.5) on an arbitrary group.

**Theorem 3.3.** Let $G$ be a group, let $\varphi : G \to G$ be an endomorphism, and let the pair $f, g : G \to \mathbb{C}$ be a solution of the functional equation (1.5). Then we have the following possibilities:

(a) $f = 0$ and $g$ is arbitrary.
(b) There exists a character $\chi$ of $G$ such that

$$f = \alpha \chi \quad \text{and} \quad g = \frac{\chi + \chi \circ \varphi}{2},$$

for some $\alpha \in \mathbb{C}\setminus\{0\}$.

(c) There exists a character $\chi$ of $G$ with $\chi = \chi \circ \varphi^2$ such that

$$g = \frac{\chi + \chi \circ \varphi}{2}.$$

Furthermore, we have:

(i) If $\chi \neq \chi \circ \varphi$, then

$$f = \alpha \chi + \beta \chi \circ \varphi,$$

for some $\alpha, \beta \in \mathbb{C}\setminus\{0\}$.

(ii) If $\chi = \chi \circ \varphi$, then there exists a non-zero additive function $A : G \to \mathbb{C}$ with $A \circ \varphi = -A$ such that

$$f = (\alpha + A)\chi,$$

for some $\alpha \in \mathbb{C}$.

Conversely, the functions given with these properties satisfy the functional equation (1.5). Moreover, if $G$ is a topological group, $f \neq 0$, and $f, g \in C(G)$, then $\chi, \chi \circ \varphi, A \in C(G)$.

The monoid version (Theorem 3.4) differs from Theorem 3.3 only when $\chi = \chi \circ \varphi$ (case (c)(ii)), where the formulations are more complicated. The conclusions of the two versions agree if $\chi$ vanishes nowhere, which is the case on groups.

**Theorem 3.4.** Let $M$ be a monoid which is generated by its squares, let $\varphi : M \to M$ be an endomorphism, and let the pair $f, g : M \to \mathbb{C}$ be a solution of the functional equation (1.5). Then we have the following possibilities:

(a) $f = 0$ and $g$ is arbitrary.

(b) There exists a non-zero multiplicative function $\chi : M \to \mathbb{C}$ such that

$$f = \alpha \chi \quad \text{and} \quad g = \frac{\chi + \chi \circ \varphi}{2},$$

for some $\alpha \in \mathbb{C}\setminus\{0\}$.

(c) There exists a non-zero multiplicative function $\chi : M \to \mathbb{C}$ with $\chi = \chi \circ \varphi^2$ such that

$$g = \frac{\chi + \chi \circ \varphi}{2}.$$

Furthermore, we have:

(i) If $\chi \neq \chi \circ \varphi$, then

$$f = \alpha \chi + \beta \chi \circ \varphi,$$

for some $\alpha, \beta \in \mathbb{C}\setminus\{0\}$.

(ii) If $\chi = \chi \circ \varphi$, then there exists a non-zero additive function $A : M \setminus I_\chi \to \mathbb{C}$ with $A \circ \varphi = -A$ such that

$$f(x) = \begin{cases} 
\alpha + A(x)\chi(x) & \text{for } x \in M \setminus I_\chi \\
0 & \text{for } x \in I_\chi
\end{cases}$$

for some $\alpha \in \mathbb{C}$.
Conversely, the functions given with these properties satisfy the functional equation (1.5).
Moreover, if $M$ is a topological monoid generated by its squares, $f \neq 0$, and $f, g \in C(M)$, then $\chi, \chi \circ \phi \in C(M)$, while $A \in C(M, M)$.

**Proof of Theorems 3.3 and 3.4.** If $f = 0$ we deal with Case (a). So during the rest of the proof we will assume that $f \neq 0$. Making the substitutions $(x, y, z)$, $(\phi(z)x, y)$ and $(x, y, z)$ in (1.5), we get respectively
\[
\begin{align*}
f(xyz) + f(\phi(z)xy) &= 2f(xy)g(z), \\
f(\phi(z)xy) + f(\phi(yz)x) &= 2f(\phi(z)x)g(y) = 2[2f(x)g(x)g(y) - f(xz)]g(y), \\
f(xyz) + f(\phi(yz)x) &= 2f(x)g(yz).
\end{align*}
\]
Subtracting the middle identity from the sum of the other two we get after some simplifications that
\[
f(xyz) - f(x)g(yz) = [f(xy) - f(x)g(y)]g(x) + [f(xz) - f(x)g(z)]g(y),
\]
i.e.
\[
h_3(yz) = h_4(y)g(z) + h_5(z)g(y),
\]
where $h_6(b) := f(ab) - f(a)g(b)$. So, the pair $(h_4, g)$ is a solution of the sine addition law for any $x \in S$. In particular for $x = e$ and in this case we have
\[
f = h_4 + f(e)g.
\]
**Case 1:** Suppose that $h_4 = 0$. Then $f = f(e)g$ and hence $f(e) \neq 0$. Indeed, $f(e) = 0$ would entail $f = 0$, contradicting our assumption. So, $g$ is a solution of the functional equation (1.4). According to Lemma 3.2, we have only the following two cases.

**Subcase 1.1:** There exists a non-zero multiplicative function $\chi : M \to \mathbb{C}$ with $\chi \circ \phi = 0$ such that $g = \frac{1}{2} \chi$. So $g = (\chi + \chi \circ \phi)/2$. It is Case (b) of our theorem.

**Subcase 1.2:** There exists a multiplicative function $\chi : M \to \mathbb{C}$ with $\chi = \chi \circ \phi^2$ such that $g = (\chi + \chi \circ \phi)/2$. Since $g \neq 0$ (because $f \neq 0$), we have $\chi \neq 0$. So we are in Case (b) or (c) (i).

**Case 2:** We now suppose that $h_4 \neq 0$. From Lemma 3.1, we see that there exist two multiplicative functions $\chi_1, \chi_2 : M \to \mathbb{C}$ such that
\[
g = \frac{\chi_1 + \chi_2}{2}.
\]

**Subcase 2.1:** Assume that $\chi_1 \neq \chi_2$. By Lemma 3.1, we have $h_4 = c(\chi_1 - \chi_2)$ for some constant $c \in \mathbb{C} \setminus \{0\}$. So
\[
\begin{align*}f &= c(\chi_1 - \chi_2) + \frac{1}{2}f(e)(\chi_1 + \chi_2) \\
&= \alpha \chi_1 + \beta \chi_2,
\end{align*}
\]
where $\alpha = c + \frac{1}{2}f(e)$ and $\beta = -c + \frac{1}{2}f(e)$. Substituting $f$ into (1.5), we find after a reduction that
\[
\alpha \chi_1(x)[\chi_2(y) - \chi_1 \circ \phi(y)] + \beta \chi_2(x)[\chi_1(y) - \chi_2 \circ \phi(y)] = 0,
\]
for all $x, y \in M$. Since $\chi_1 \neq \chi_2$ we get that
\[
\begin{align*}
\left\{ \begin{array}{l}
\alpha \chi_1(x)[\chi_2(y) - \chi_1 \circ \phi(y)] = 0 \\
\beta \chi_2(x)[\chi_1(y) - \chi_2 \circ \phi(y)] = 0
\end{array} \right.
\end{align*}
\]
(3.1)
If $\chi_2 = 0$, then $\chi_1 \neq 0$ and hence $\alpha \chi_1 \circ \phi = 0$. If $\alpha = 0$, then $f = 0$ but $f \neq 0$ by assumption, so that $\alpha \neq 0$ and hence $\chi_1 \circ \phi = 0$. Thus $f = \alpha \chi_1$ and $g = \frac{1}{2} \chi_1$. So we are in Case (c) (b). The same is true for $\chi_1 = 0$ and $\chi_2 \neq 0$. If $\chi_1 \neq 0$ and $\chi_2 \neq 0$. Then (3.1) becomes
\[
\begin{align*}
\left\{ \begin{array}{l}
\alpha [\chi_2(y) - \chi_1 \circ \phi(y)] = 0 \\
\beta [\chi_1(y) - \chi_2 \circ \phi(y)] = 0
\end{array} \right.
\end{align*}
\]
(3.2)
Suppose that $\beta = 0$. Then $\alpha = 2c \neq 0$. From (3.2), we see that $\chi_2 = \chi_1 \circ \phi$ and arrive at the solution in Case (b) with $\chi = \chi_1$. The same is true for $\alpha = 0$ with the multiplicative function $\chi_2$ replacing $\chi$.  

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We now suppose that $\alpha \neq 0$ and $\beta \neq 0$. From (3.2), we see that $\chi_2 = \chi_1 \circ \varphi$ and $\chi_2 \circ \varphi = \chi_1$. So $\chi_1 \neq \chi_1 \circ \varphi$ (because $\chi_1 \neq \chi_2$), $\chi_1 = \chi_2 \circ \varphi = \chi_1 \circ \varphi^2$, and we have
\[
f = \alpha \chi_1 + \beta \chi_2 = \alpha \chi_1 + \beta \chi_1 \circ \varphi,
g = \frac{\chi_1 + \chi_2}{2} = \frac{\chi_1 + \chi_1 \circ \varphi}{2}.
\]
This is Case (c)(i) with $\chi = \chi_1$.

**Subcase 2.2:** Assume that $\chi_1 = \chi_2 = \chi$. If $M$ is a group, we get from Lemma 3.1 that $h_c = \chi A$ for some additive function $A : M \to \mathbb{C}$. So
\[
g(\chi) = \chi \quad \text{and} \quad f = \chi A + f(\epsilon) \chi = (\alpha + A) \chi,
\]
where $\alpha = f(\epsilon)$. Substituting $f$ into (1.5), we get
\[
\left[ A(y) - A(x) - \alpha \varphi(y) + [A \circ \varphi(y) + A(x)] + \alpha \right] \chi \circ \varphi(y) = 0,
\]
for all $x, y \in M$. Using (3.3) and the fact that $A \neq 0$ (because $h_e \neq 0$), we infer that $\chi = \chi \circ \varphi$ and $A \circ \varphi = -A$. So, we are in case (c)(ii) of Theorem 3.3.

If $M$ is a monoid which is generated by its squares, we get from Lemma 3.1 that there exists an additive function $A : M \setminus I \to \mathbb{C}$ for which
\[
h_c(x) = \begin{cases} 
\chi(x)A(x) & \text{for } x \in M \setminus I \\
0 & \text{for } x \in I
\end{cases}
\]
Since $f = h_c + f(\epsilon)g$ and $\chi(x) = 0$ for $x \in I$, we have
\[
f(x) = \begin{cases} 
(\alpha + A(x))\chi(x) & \text{for } x \in M \setminus I \\
0 & \text{for } x \in I
\end{cases}
\]
where $\alpha = f(\epsilon)$. Substitution into (1.5) gives that $\chi = \chi \circ \varphi$ and $A \circ \varphi = -A$. So, we are in case (c)(ii) of Theorem 3.4.

Conversely, simple computations prove that the formulas above for $f$ and $g$ define solutions of (1.5).

The continuity statements follow from Lemma 3.1 and [13, Theorem 3.18(d)].

**Remark 3.5.** When the condition $\chi = \chi \circ \varphi^2$ in Case (c) of Theorem 3.3 is always satisfied, the Cases (b) and (c) of Theorem 3.3 can be reduced to the following:

(d) There exists a character $\chi$ of $G$ such that
\[
g = \frac{\chi + \chi \circ \varphi}{2}.
\]
Furthermore, we have:

(i) If $\chi \neq \chi \circ \varphi$, then
\[
f = \alpha \chi + \beta \chi \circ \varphi,
\]
for some $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$.

(ii) If $\chi = \chi \circ \varphi$, then there exists an additive function $A : G \to \mathbb{C}$ with $A \circ \varphi = -A$ such that
\[
f = (\alpha + A) \chi,
\]
for some $\alpha \in \mathbb{C}$.

The same idea is valid for Theorem 3.4.
4 Applications

As immediate consequences of Theorems 3.3 and 3.4, combined with Remark 3.5, we obtain the following two corollaries.

**Corollary 4.1** ([5], Theorem 3.6). Let $G$ be a group, let $\sigma : G \to G$ be an involutive automorphism, and let the pair $f, g : G \to \mathbb{C}$ be a solution of the variant of Wilson's functional equation

$$f(xy) + f(\sigma(y)x) = 2f(x)g(y), \quad x, y \in G. \quad (4.1)$$

Then we have the following possibilities:

(a) $f = 0$ and $g$ is arbitrary.

(b) There exists a character $\chi$ of $G$ such that

$$g = \chi + \chi \circ \sigma.$$

Furthermore, we have:

(i) If $\chi \neq \chi \circ \sigma$, then

$$f = \alpha \chi + \beta \chi \circ \sigma,$$

for some $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$.

(ii) If $\chi = \chi \circ \sigma$, then there exists an additive function $A : G \to \mathbb{C}$ with $A \circ \sigma = -A$ such that

$$f = (\alpha + A)\chi,$$

for some $\alpha \in \mathbb{C}$.

Conversely, the functions given with these properties satisfy the functional equation (4.1). Moreover, if $G$ is a topological group, $f \neq 0$, and $f, g \in C(G)$, then $\chi, \chi \circ \sigma, A \in C(G)$.

**Corollary 4.2** ([5], Theorem 3.7). Let $M$ be a monoid which is generated by its squares, let $\sigma : M \to M$ be an involutive automorphism, and let the pair $f, g : M \to \mathbb{C}$ be a solution of the functional equation (4.1). Then we have the following possibilities:

(a) $f = 0$ and $g$ is arbitrary.

(b) There exists a non-zero multiplicative function $\chi : M \to \mathbb{C}$ such that

$$g = \frac{\chi + \chi \circ \sigma}{2}.$$

Furthermore, we have:

(i) If $\chi \neq \chi \circ \sigma$, then

$$f = \alpha \chi + \beta \chi \circ \sigma,$$

for some $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0,0)\}$.

(ii) If $\chi = \chi \circ \sigma$, then there exists an additive function $A : M \setminus I_\chi \to \mathbb{C}$ with $A \circ \sigma = -A$ such that

$$f(x) = \begin{cases} (\alpha + A(x))\chi(x) & \text{for } x \in M \setminus I_\chi \\ 0 & \text{for } x \in I_\chi \end{cases}$$

for some $\alpha \in \mathbb{C}$.

Conversely, the functions given with these properties satisfy the functional equation (4.1).
Moreover, if $M$ is a topological monoid generated by its squares, $f \neq 0$, and $f, g \in C(M)$, then $\chi, \chi \circ \sigma \in C(M)$, while $A \in C(M \setminus I_\chi)$.  

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In the rest of the paper let $G$ be a group, $\phi : G \to G$ be an endomorphism, and let $z_0 \in G$ be a fixed element. In view of Theorem 3.3, we obtain the following result which is an extension of the result of [13, Exercise 11.6].

**Corollary 4.3.** Let the pair $f, g : G \to \mathbb{C}$ be a solution of the functional equation

\[
f(xy_0) + f(\phi(y)x_0) = 2f(x)g(y), \quad x,y \in G.
\]  \hspace{1cm} (4.2)

Then we have the following possibilities:

(a) $f = 0$ and $g$ is arbitrary.

(b) There exists a character $\chi$ of $G$ such that

\[
f = \alpha \chi \quad \text{and} \quad g = \frac{\chi(z_0)}{2}(\chi + \chi \circ \phi),
\]

for some $\alpha \in \mathbb{C}\setminus\{0\}$.

(c) There exists a character $\chi$ of $G$ with $\chi = \chi \circ \phi^2$ and $\chi \circ \phi(z_0) = \chi(z_0)$ such that

\[
g = \frac{\chi(z_0)}{2}(\chi + \chi \circ \phi).
\]

Furthermore, we have:

(i) If $\chi \neq \chi \circ \phi$, then

\[
f = \alpha \chi + \beta \chi \circ \phi,
\]

for some $\alpha, \beta \in \mathbb{C}\setminus\{0\}$.

(ii) If $\chi = \chi \circ \phi$, then there exists a non-zero additive function $A : G \to \mathbb{C}$ with $A \circ \phi = -A$ and $A(z_0) = 0$ such that

\[
f = (\alpha + A)\chi,
\]

for some $\alpha \in \mathbb{C}$.

Conversely, the functions given with these properties satisfy the functional equation (4.2). Moreover, if $G$ is a topological group, $f \neq 0$, and $f, g \in C(G)$, then $\chi, \chi \circ \phi, A \in C(G)$.

**Proof.** If $f = 0$, then (a) is the case. From now on we assume that $f \neq 0$. By putting $y = e$ in (4.2) we get that

\[
f(xz_0) = g(e)f(x), \quad x \in G.
\]  \hspace{1cm} (4.3)

Since $f \neq 0$, we get immediately $g(e) \neq 0$. So, using (4.3), we can reformulate the form of Eq. (4.2) as

\[
f(xy) + f(\phi(y)x) = 2f(x)\frac{g(y)}{g(e)}, \quad x,y \in G.
\]

So the pair $(f, \frac{g}{g(e)})$ is a solution of (1.5). Since $f \neq 0$, we know from Theorem 3.3 that there are only the following three cases:

**Case 1:** There exist a character $\chi$ of $G$ and a constant $\alpha \in \mathbb{C}\setminus\{0\}$ such that

\[
f = \alpha \chi \quad \text{and} \quad g = \frac{g(e)}{2}(\chi + \chi \circ \phi).
\]

Simple computations based on (4.3) shows that $g(e) = \chi(z_0)$. So we are in Case (b) of our statement.

**Case 2:** There exist a character $\chi$ of $G$ with $\chi \neq \chi \circ \phi$ and $\chi = \chi \circ \phi^2$, and constants $\alpha, \beta \in \mathbb{C}\setminus\{0\}$ such that

\[
f = \alpha \chi + \beta \chi \circ \phi \quad \text{and} \quad g = \frac{g(e)}{2}(\chi + \chi \circ \phi).
\]

Simple computations based on (4.3) shows that

\[
\alpha[g(e) - \chi(z_0)]\chi + \beta[g(e) - \chi \circ \phi(z_0)]\chi \circ \phi = 0.
\]

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By the linear independence of different characters we infer that
\[
\begin{cases}
\alpha [g(e) - \chi(z_0)] = 0 \\
\beta [g(e) - \chi \circ \varphi(z_0)] = 0
\end{cases}
\]
Since \(\alpha, \beta \neq 0\) we get that \(g(e) = \chi(z_0) = \chi \circ \varphi(z_0)\). So we arrive at the solution in Case (c)(i).

**Case 3:** There exist a character \(\chi\) of \(G\) with \(\chi = \chi \circ \varphi\), an additive function \(A: G \to \mathbb{C}\) with \(A \circ \varphi = -A\), and a constant \(\alpha \in \mathbb{C}\) such that
\[
f = (\alpha + A)\chi \quad \text{and} \quad g = g(e)\chi.
\]

Using (4.3), we get that
\[
[g(e) - \chi(z_0)]A = A(z_0)\chi(z_0) - \alpha [g(e) - \chi(z_0)].
\]
Since \(A\) is additive, the last equality can hold only if \(g(e) = \chi(z_0)\) and \(A(z_0) = 0\). So we are in Case (b) or (c)(ii). This finishes the necessity assertion.

Conversely, simple computations prove that the formulas above for \((f, g)\) define solutions of (4.2).

The continuity statements follow from Theorem 3.3.

As another consequence of Theorem 3.3, we have the following result.

**Corollary 4.4.** The solutions \(f, g: G \to \mathbb{C}\) of the functional equation
\[
f(xyz_0) + f(\varphi(y)xz_0) = 2g(x)f(y), \quad x, y \in G,
\]
are the following:

(a) \(f = 0\) and \(g\) is arbitrary.

(b) There exists a character \(\chi\) of \(G\) and a constant \(c \in \mathbb{C}\setminus\{0\}\) such that
\[
f = c \frac{\chi + \chi \circ \varphi}{2} \quad \text{and} \quad g = \frac{\chi(z_0)}{2} \chi + \frac{\chi \circ \varphi(z_0)}{2} \chi \circ \varphi.
\]

**Proof.** We leave out the simple verifications that the formulas of (a) and (b) define solutions of (4.4). It is thus smug to prove that any solution \((f, g)\) of (4.4) falls into one of these two categories. We note that the proof is similar to the proof of Corollary 4.3.

The first case is obvious, so we suppose that \(f \neq 0\). By putting \(y = e\) in (4.4) we get that
\[
f(xz_0) = f(e)g(x), \quad x \in G.
\]
Since \(f \neq 0\), we have \(f(e) \neq 0\). So, using (4.5), we can reformulate the form of Eq. (4.4) as
\[
g(xy) + g(\varphi(y)x) = 2g(x)\frac{f(y)}{f(e)}, \quad x, y \in G.
\]

So the pair \((g, \frac{f}{f(e)})\) is a solution of (1.5). Since \(f \neq 0\), we know from Theorem 3.3 that there are only the following two cases:

**Case 1:** There exists a character \(\chi\) of \(G\) such that
\[
g = \alpha \chi \quad \text{and} \quad f = \frac{f(e)}{2}(\chi + \chi \circ \varphi),
\]
for some \(\alpha \in \mathbb{C}\setminus\{0\}\).

Simple computations based on (4.5) shows that
\[
|\chi(z_0) - 2\alpha|\chi + \chi \circ \varphi(z_0)\chi \circ \varphi = 0.
\]
Since $\chi \circ \varphi (z_0) \neq 0$, the last equality can only holds if $\chi = \chi \circ \varphi$ and $\alpha = \chi (z_0)$. So we are in Case $(b)$ above with $c = f(e)$.

**Case 2:** There exist a character $\chi$ of $G$ with $\chi \circ \varphi^2 = \chi$ and $\chi \neq \chi \circ \varphi$, and constants $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ such that

$$f = \frac{f(e)}{2} (\chi + \chi \circ \varphi) \quad \text{and} \quad g = \alpha \chi + \beta \chi \circ \varphi.$$

Simple computations based on (4.5) shows that

$$|\chi(z_0) - 2\alpha\chi + [\chi \circ \varphi(z_0) - 2\beta]\chi \circ \varphi| = 0.$$

By the linear independence of different characters we infer that $\alpha = \frac{1}{2} \chi(z_0)$ and $\beta = \frac{1}{2} \chi \circ \varphi(z_0)$. So we are in Case $(b)$ above with $c = f(e)$.

**Case 3:** There exist a character $\chi$ of $G$ with $\chi = \chi \circ \varphi$, a non-zero additive function $A : G \to \mathbb{C}$ with $A \circ \varphi = -A$, and a constant $\alpha \in \mathbb{C}$ such that

$$f = f(e) \chi \quad \text{and} \quad g = (\alpha + A) \chi.$$

Using (4.5), we get that

$$\alpha + A = \chi(z_0).$$

Since $A$ is additive, the last equality can only holds if $A = 0$ and $\alpha = \chi(z_0)$. This case does not apply, because $A \neq 0$ by assumption. This finishes the proof.

As a consequence of Corollary 4.4, we have the following result which is a natural extension of Kannappan’s functional equation (1.3).

**Corollary 4.5.** The solutions $f : G \to \mathbb{C}$ of the functional equation

$$f(xy_0) + f(\varphi(y)x_0) = 2f(x)f(y), \quad x, y \in G,$$

are either $f \equiv 0$ or

$$f = \chi(z_0) \frac{\chi + \chi \circ \varphi}{2},$$

where $\chi$ is a character of $G$ satisfying $\chi \circ \varphi(z_0) = \chi(z_0)$.

**Remark 4.6.** By using Theorem 3.4, we can get the solutions of the functional equations (4.2), (4.4), and (4.6) on monoids that are generated by their squares.

**References**


