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Propagation of acoustic wave's motion in orthotropic Cylinders of infinite length

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Abstract
We report, in the present work, a numerical method for investigating guided waves propagation in a homogeneous infinite cylinder composed of elastic material. This method makes use of Legendre polynomials series and harmonic function to express different displacement components which are introduced into the equation of motion. The advantage of this method is the possibility to incorporate the stress-free boundary conditions directly into the equations of motion by assuming position-dependent elastic constants and mass density. The solution of the wave equations can be reduced to an eigenvalue problem. Numerical results are presented and compared with those published earlier in order to validate our polynomial approach. For certain specific modes, dispersion curves and field profiles such as mechanical displacements, normal stresses are presented. The developed software is capable of dealing efficiently and accurately with a variety of homogeneous and inhomogeneous cylinders.
Keywords: Guided waves, elastic material, wave propagation, polynomial approach.

I. Introduction
Over the past three decades, the study of guided waves propagation have received considerable attention and research effort with their increasing usage in various applications including the nondestructive testing (NDT), structural health monitoring (SHM) and used in many other engineering fields. New faster, more sensitive and more economical ways of looking at materials and structures have become possible when compared to the previously used normal beam ultrasonic or other inspection techniques [1].
The orthogonal polynomial approach was used, for the first time, to study the propagation of acoustic waves in homogeneous semi-infinite wedges [2]. After that, the method was extended to simulate the propagation of guided waves in cylindrically orthotropic material. The exact solutions to the equation of motion, for infinitely long isotropic cylinders, was developed by Pochhamer [3] and Chree [4]. For solving vibrational problems, Mirsky [5] was the first to study axisymmetric free vibration for orthotropic cylindrical shells with infinite length based en

three-dimensional elastic theory. Recently, a similar formulation has been used to analyze the behavior of wave propagation in inhomogeneous cylinders [6-7]. Also, Jiangong and al [8] were extended the polynomial method to simulate wave propagation in the circumferential direction of general multilayered plates composed of piezoelectric material.

In this context, we describe in this paper the polynomial approach applicated to investigate the behavior of wave propagation in an orthotropic homogeneous cylinder with infinite length.

2. Model description and formulation of the problem

Let us consider a cylindrically structure with infinite length as shown in Fig.1.

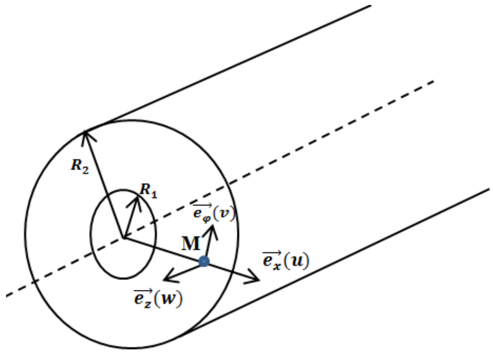

Figure 1. Schematic structure of studied cylinder

The entire problem will be dealt with the system of cylindrical coordinates $\emptyset z$. R_1 and R_2 are the inner and outer radius respectively, H represent the thickness.

Under the assumption of small deformations, the strain-displacement relations in terms of cylindrical coordinate system are expressed by [8].

$$\begin{aligned} S_{rr} &= \frac{\partial u}{\partial r} & 2S_{\phi z} &= \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \phi} \\ S_{\phi\phi} &= \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \phi} & 2S_{rz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r} \\ S_{zz} &= \frac{\partial w}{\partial z} & 2S_{r\phi} &= \frac{1}{r}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial r} - \frac{v}{r} \end{aligned} \tag{1}$$

For an orthotropic material, the stress components are related to the displacement components by the Hook's law expressed as follow:

$$\begin{aligned}
T_{rr} &= C_{11}\frac{\partial u}{\partial r} + C_{12}\left(\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \phi}\right) + C_{13}\frac{\partial w}{\partial z} + 2C_{14}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \phi}\right) + 2C_{15}\left(\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + 2C_{16}\left(\frac{1}{r}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial r} - \frac{v}{r}\right) \\
T_{\phi\phi} &= C_{21}\frac{\partial u}{\partial r} + C_{22}\left(\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \phi}\right) + C_{23}\frac{\partial w}{\partial z} + 2C_{24}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \phi}\right) + 2C_{25}\left(\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + 2C_{26}\left(\frac{1}{r}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial r} - \frac{v}{r}\right) \\
T_{zz} &= C_{31}\frac{\partial u}{\partial r} + C_{32}\left(\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \phi}\right) + C_{33}\frac{\partial w}{\partial z} + 2C_{34}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \phi}\right) + 2C_{35}\left(\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + 2C_{36}\left(\frac{1}{r}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial r} - \frac{v}{r}\right) \\
T_{\phi z} &= C_{41}\frac{\partial u}{\partial r} + C_{42}\left(\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \phi}\right) + C_{43}\frac{\partial w}{\partial z} + 2C_{44}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \phi}\right) + 2C_{45}\left(\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + 2C_{46}\left(\frac{1}{r}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial r} - \frac{v}{r}\right) \\
T_{rz} &= C_{51}\frac{\partial u}{\partial r} + C_{52}\left(\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \phi}\right) + C_{53}\frac{\partial w}{\partial z} + 2C_{54}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \phi}\right) + 2C_{55}\left(\frac{\partial u}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial r}\right) + 2C_{56}\left(\frac{1}{r}\frac{\partial u}{\partial \phi} + \frac{\partial v}{\partial r} - \frac{v}{r}\right)
\end{aligned} \tag{2}$$

$$T_{r\phi}=C_{61}\frac{\partial u}{\partial r}+C_{62}\left(\frac{u}{r}+\frac{1}{r}\frac{\partial v}{\partial \phi}\right)+C_{63}\frac{\partial w}{\partial z}+2C_{64}\left(\frac{\partial v}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial \phi}\right)+2C_{65}\left(\frac{\partial u}{\partial z}+\frac{1}{r}\frac{\partial w}{\partial r}\right)+2C_{66}\left(\frac{1}{r}\frac{\partial u}{\partial \phi}+\frac{\partial v}{\partial r}-\frac{v}{r}\right)$$

The field equation governing wave propagation is given by the field of Newton's low which can be expressed as [9]:

$$\nabla.T(M)=\rho\frac{\partial^2\vec{U}(M,t)}{\partial t^2} \tag{3}$$

$$\begin{aligned}
&\frac{\partial T_{rr}}{\partial r}+\frac{\partial T_{rz}}{\partial z}+\frac{1}{r}\frac{\partial T_{r\phi}}{\partial \phi}+\frac{T_{rr}-T_{\phi\phi}}{r}=\rho\frac{\partial^2 u}{\partial t^2}\\
&\frac{\partial T_{r\phi}}{\partial r}+\frac{\partial T_{\phi z}}{\partial z}+\frac{1}{r}\frac{\partial T_{\phi\phi}}{\partial \phi}+\frac{2T_{r\phi}}{r}=\rho\frac{\partial^2 v}{\partial t^2}\\
&\frac{\partial T_{rz}}{\partial r}+\frac{\partial T_{zz}}{\partial z}+\frac{1}{r}\frac{\partial T_{\phi z}}{\partial \phi}+\frac{T_{rz}}{r}=\rho\frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{4}$$

Throughout this paper, we adopt variables: $q_1 = kr$, $q_2 = \phi$, $q_3 = kz$ which are introduced for mathematical convenience, in the equation of motion. Where k represent the magnitude of the wave vector.

$$\begin{aligned}
\frac{T_{rr}}{k}&=C_{11}\frac{\partial u}{\partial q_1}+C_{12}\left(\frac{u}{q_1}+\frac{1}{q_1}\frac{\partial v}{\partial q_2}\right)+C_{13}\frac{\partial w}{\partial q_3}+2C_{14}\left(\frac{\partial v}{\partial q_3}+\frac{1}{q_1}\frac{\partial w}{\partial q_2}\right)+2C_{15}\left(\frac{\partial u}{\partial q_3}+\frac{1}{q_1}\frac{\partial w}{\partial q_1}\right)+2C_{16}\left(\frac{1}{q_1}\frac{\partial u}{\partial q_2}+\frac{\partial v}{\partial q_1}-\frac{v}{q_1}\right)\\
\frac{T_{\phi\phi}}{k}&=C_{21}\frac{\partial u}{\partial q_1}+C_{22}\left(\frac{u}{q_1}+\frac{1}{q_1}\frac{\partial v}{\partial q_2}\right)+C_{23}\frac{\partial w}{\partial q_3}+2C_{24}\left(\frac{\partial v}{\partial q_3}+\frac{1}{q_1}\frac{\partial w}{\partial q_2}\right)+2C_{25}\left(\frac{\partial u}{\partial q_3}+\frac{1}{q_1}\frac{\partial w}{\partial q_1}\right)+2C_{26}\left(\frac{1}{q_1}\frac{\partial u}{\partial q_2}+\frac{\partial v}{\partial q_1}-\frac{v}{q_1}\right)\\
\frac{T_{zz}}{k}&=C_{31}\frac{\partial u}{\partial q_1}+C_{32}\left(\frac{u}{q_1}+\frac{1}{q_1}\frac{\partial v}{\partial q_2}\right)+C_{33}\frac{\partial w}{\partial q_3}+2C_{34}\left(\frac{\partial v}{\partial q_3}+\frac{1}{q_1}\frac{\partial w}{\partial q_2}\right)+2C_{35}\left(\frac{\partial u}{\partial q_3}+\frac{1}{q_1}\frac{\partial w}{\partial q_1}\right)+2C_{36}\left(\frac{1}{q_1}\frac{\partial u}{\partial q_2}+\frac{\partial v}{\partial q_1}-\frac{v}{q_1}\right)\\
\frac{T_{\phi z}}{k}&=C_{41}\frac{\partial u}{\partial q_1}+C_{42}\left(\frac{u}{q_1}+\frac{1}{q_1}\frac{\partial v}{\partial q_2}\right)+C_{43}\frac{\partial w}{\partial q_3}+2C_{44}\left(\frac{\partial v}{\partial q_3}+\frac{1}{q_1}\frac{\partial w}{\partial q_2}\right)+2C_{45}\left(\frac{\partial u}{\partial q_3}+\frac{1}{q_1}\frac{\partial w}{\partial q_1}\right)+2C_{46}\left(\frac{1}{q_1}\frac{\partial u}{\partial q_2}+\frac{\partial v}{\partial q_1}-\frac{v}{q_1}\right)\\
\frac{T_{rz}}{k}&=C_{51}\frac{\partial u}{\partial q_1}+C_{52}\left(\frac{u}{q_1}+\frac{1}{q_1}\frac{\partial v}{\partial q_2}\right)+C_{53}\frac{\partial w}{\partial q_3}+2C_{54}\left(\frac{\partial v}{\partial q_3}+\frac{1}{q_1}\frac{\partial w}{\partial q_2}\right)+2C_{55}\left(\frac{\partial u}{\partial q_3}+\frac{1}{q_1}\frac{\partial w}{\partial q_1}\right)+2C_{56}\left(\frac{1}{q_1}\frac{\partial u}{\partial q_2}+\frac{\partial v}{\partial q_1}-\frac{v}{q_1}\right)\\
\frac{T_{r\phi}}{k}&=C_{61}\frac{\partial u}{\partial q_1}+C_{62}\left(\frac{u}{q_1}+\frac{1}{q_1}\frac{\partial v}{\partial q_2}\right)+C_{63}\frac{\partial w}{\partial q_3}+2C_{64}\left(\frac{\partial v}{\partial q_3}+\frac{1}{q_1}\frac{\partial w}{\partial q_2}\right)+2C_{65}\left(\frac{\partial u}{\partial q_3}+\frac{1}{q_1}\frac{\partial w}{\partial q_1}\right)+2C_{66}\left(\frac{1}{q_1}\frac{\partial u}{\partial q_2}+\frac{\partial v}{\partial q_1}-\frac{v}{q_1}\right)
\end{aligned} \tag{5}$$

The field of Newton's law becomes:

$$\begin{aligned}
&\frac{\partial T_{rr}}{\partial q_1}+\frac{\partial T_{rz}}{\partial q_3}+\frac{1}{q_1}\frac{\partial T_{r\phi}}{\partial q_2}+\frac{T_{rr}-T_{\phi\phi}}{q_1}=\frac{\rho}{k}\frac{\partial^2 u}{\partial t^2}\\
&\frac{\partial T_{r\phi}}{\partial q_1}+\frac{\partial T_{\phi z}}{\partial q_3}+\frac{1}{q_1}\frac{\partial T_{\phi\phi}}{\partial q_2}+\frac{2T_{r\phi}}{q_1}=\frac{\rho}{k}\frac{\partial^2 v}{\partial t^2}\\
&\frac{\partial T_{rz}}{\partial q_1}+\frac{\partial T_{zz}}{\partial q_3}+\frac{1}{q_1}\frac{\partial T_{\phi z}}{\partial q_2}+\frac{T_{rz}}{q_1}=\frac{\rho}{k}\frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{6}$$

In order to obtain the dispersion relations for cylinder modes, we use the position-dependence of elastic constants and mass density.

The cylindrical form is defined by following equations:

$$kR_1 \le q_1 \le kR_2 \qquad 0 \le q_2 \le 2\pi \qquad -\infty \le q_3 \le +\infty \tag{7}$$

The position dependence of the elastic constants and mass density is obtained by:

$$C_{ij}(q_1) = C_{ij}\,\pi(kR_1, kR_2)$$
$$\rho(q_1) = \rho\,\pi(kR_1, kR_2) \tag{8}$$

Where C_{ij} are the elastic coefficients.

The rectangular window function $\pi(kR_1, kR_2)$ is defined by:

$$\pi(kR_1, kR_2) = \begin{cases} 1 & kR_1 \le q_1 \le kR_2 \\ 0 & Elsewhere \end{cases} \tag{9}$$

The stress free boundary conditions expressed by (10) are directly incorporated in the constitutive equations of our problem:

$$T_{rr}(q_1 = kR_1, kR_2) = T_{rz}(q_1 = kR_1, kR_2) = T_{r\phi}(q_1 = kR_1, kR_2) = 0 \tag{10}$$

The field of equation becomes:

$$\frac{\partial[T_{rr}\pi(q_1)]}{\partial q_1} + \frac{\partial[T_{rz}\pi(q_1)]}{\partial q_3} + \frac{1}{q_1}\frac{\partial[T_{r\phi}\pi(q_1)]}{\partial q_2} + \frac{T_{rr}\pi(q_1) - T_{\phi\phi}\pi(q_1)}{q_1} = \frac{\rho}{k}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial[T_{r\phi}\pi(q_1)]}{\partial q_1} + \frac{\partial T_{\phi z}}{\partial q_3} + \frac{1}{q_1}\frac{\partial T_{\phi\phi}}{\partial q_2} + \frac{2[T_{r\phi}\pi(q_1)]}{q_1} = \frac{\rho}{k}\frac{\partial^2 v}{\partial t^2} \quad \frac{\partial[T_{rz}\pi(q_1)]}{\partial q_1} + \frac{\partial T_{zz}}{\partial q_3} + \frac{1}{q_1}\frac{\partial T_{\phi z}}{\partial q_2} + \frac{[T_{rz}\pi(q_1)]}{q_1} = \frac{\rho}{k}\frac{\partial^2 w}{\partial t^2} \tag{11}$$

In the case of wave propagation in axial direction, the displacement components are assumed to be in the following form:

$$u(q_1, q_2, q_3, t) = \frac{1}{\sqrt{2\pi}} e^{inq_2} e^{i(wt-q_3)} \sum_{m=0}^{\infty} P1_m\, Q_m(q_1) \quad v(q_1, q_2, q_3, t) = \frac{1}{\sqrt{2\pi}} e^{inq_2} e^{i(wt-q_3)} \sum_{m=0}^{\infty} P2_m\, Q_m(q_1)$$

$$w(q_1, q_2, q_3, t) = \frac{1}{\sqrt{2\pi}} e^{inq_2} e^{i(wt-q_3)} \sum_{m=0}^{\infty} P3_m\, Q_m(q_1) \tag{12}$$

Where n is the number of circumferential wave, $Pi_m (i = 1,2,3)$ is the expansion coefficient and ω is the angular frequency.

$$Q_m(q_1) = \sqrt{\frac{2m+1}{kH}}\, P_m\left(\frac{2q_1 - S}{kH}\right)$$

$$S = kR_1 + kR_2$$

Where P_m is the mth Legendre polynomial. For the convergence requirement, we assume that the summation over the polynomials in (11) is halted at some finite value m=M when higher order terms are negligible.

Substituting the equations (5) and (12) into (11) , and taking the condition of the orthonormality of $Q_m(q_1)$,we obtain a form of eigenvalue problem:

$$\begin{bmatrix} A1_{mj} & B1_{mj} & C1_{mj} \\ D1_{mj} & E1_{mj} & F1_{mj} \\ G1_{mj} & H1_{mj} & I1_{mj} \end{bmatrix} \begin{Bmatrix} P1_m \\ P2_m \\ P3_m \end{Bmatrix} = -\lambda^2 \begin{bmatrix} M1_{mj} & 0 & 0 \\ 0 & M2_{mj} & 0 \\ 0 & 0 & M3_{mj} \end{bmatrix} \begin{Bmatrix} P1_m \\ P2_m \\ P3_m \end{Bmatrix} \tag{13}$$

Where $A1_{mj}, B1_{mj} \dots I1_{mj}$ and $Mi_{mj} (i = 1,2,3)$ are the elements of a non-symmetric matrix. The eigenvalue $\boldsymbol{\lambda^2}$ allow the guided wave velocity and eigenvectors $Pi_m (i = 1,2,3)$ represent the components of the particle displacement to be calculated.

III. Numerical Results

In order to validate our approach with an orthotropic material, we have been reduced the matrix of elastic constant into 9 elements:

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{14}$$

We set our simulation with a solid cylinder composed of aluminum as studied material, the program of testing method, based on foregoing formulation, has been written by Matlab.

Normalized frequency can be expressed by: $\Omega = \dfrac{wH}{C_s}$ $C_s = \left(\dfrac{C_{44}}{\rho}\right)^{0.5}$ where C_{44} and ρ are the elastic constant and mass density of the cylinder.

The dispersion behavior of waves is presented by figures 2, 3, 4 and 5. Where figure 2 represent normalized frequencies Ω for longitudinal and torsional modes as function of kH for a homogenous solid aluminum cylinder. The figure 3 represent the distribution of normalized frequencies of flexural modes F(1,i=1,2,...n) as function of kH for the same cylinder. The figure 4 shows the dispersion curves of longitudinal and torsional modes as function of fH and the figure 5 represent the dispersion curves of flexural modes as function of fH for solid aluminum homogeneous cylinder.

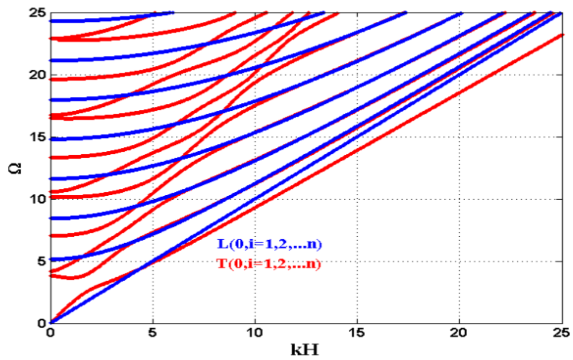

Fig 2. Normalized frequencies Ω as function of kH; L(0,i=1,2,...n) Longitudinal modes and T(0,i=1,2,...n) Torsional modes

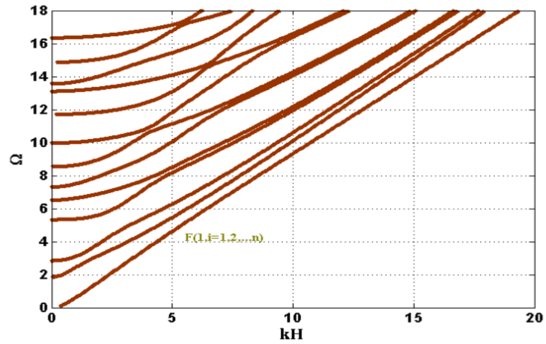

Fig 3. Normalized frequencies Ω of flexural modes F(1,i=1,2,...n) as function of kH for solid aluminum cylinder

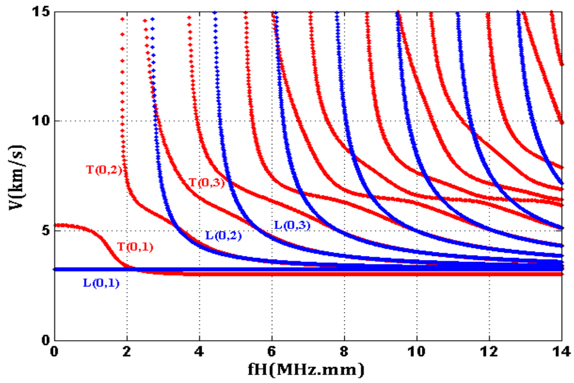

Fig 4. Dispersion curves of Longitudinal L(0,i=1,2,...n) and Torsional T(0,i=1,2,...n) modes as function of fH for aluminum homogeneous cylinder

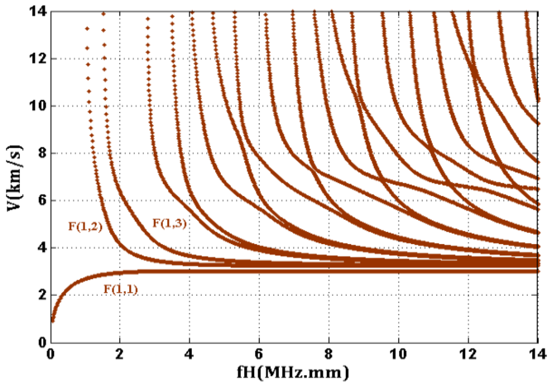

Fig 5. Dispersion curves of Flexural F(1,i=1,2,...n) modes as function of fH for solid aluminum homogeneous cylinder

In Figure 2 and 3, we find a fundamental modes without cut-off frequency L(0,1), T(0,1) and F(1,1) whereas all higher modes begin in a cut-off frequency. In the Figure 4, we observe that the first longitudinal mode L(0,1) exists and it's non dispersive at all frequencies.
Moreover, and in order to illustrate the methodologies of our approach, the different field profiles for longitudinal, torsional and flexural modes were also analyzed. The figure 6 explains the dispersion of field profiles, for different values of normalized frequencies Ω and $kH = 3.1416$, in the thickness direction for solid aluminum homogeneous cylinder. (a) Longitudinal mode $\Omega = 16.663$, (b) Torsional mode $\Omega = 33.8623$ and (c-d) Flexural modes $\Omega = 4.9175$ and $\Omega = 16.4919$ respectively.

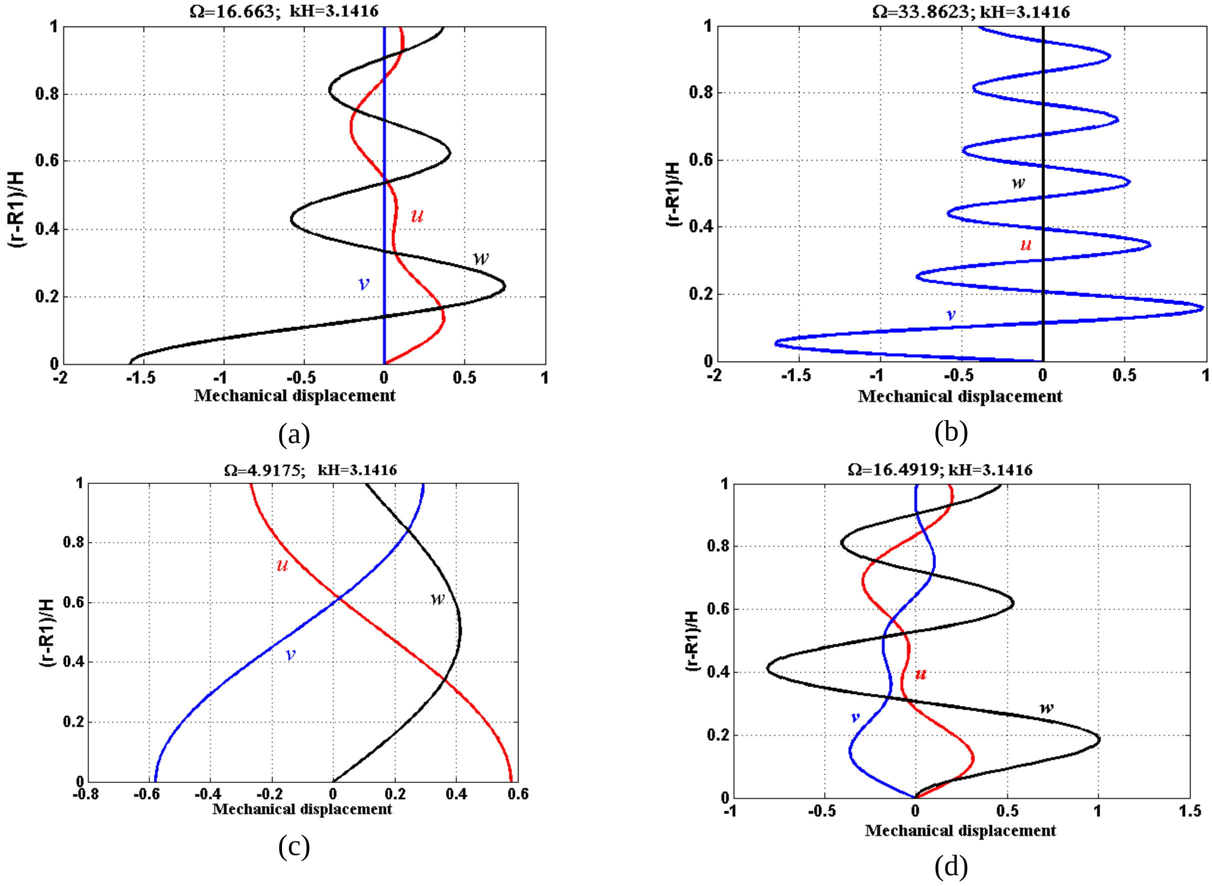

Figure 6. Field profiles in the thickness direction for solid aluminum homogeneous cylinder (a) Longitudinal mode, (b) Torsional modes and (c-d) Flexural modes.

As it shown in those representations, the component of motion in the circumferential direction is equal zero for longitudinal modes, whereas the components of motion in radial and axial directions are null for the torsional modes.

IV. Conclusion

In this paper, we have presented a model of polynomial approach in order to analyze the behavior of guided wage propagation along a solid cylinder composed of elastic material. Incorporation of the displacement components and boundary conditions directly into the equation of motion and reducing the entire problem to an eigenvalue problem allows the polynomial approach to be used widely for typical simulations.
The results obtained by formulation developed previously improve that the orthogonal polynomial method can, successfully and accurately, solve the guided waves propagation in orthotropic cylinders with infinite length.

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