Phase Space Topology and Bifurcation of Liouville Torii in the Goryatchev-Tchaplygin Top

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The classical problem of a rigid body with a fixed point is considered in the case of Goryatchev-Tchaplygin .we give a complete description of its real phase space topology. All generic bifurcation of Liouville Torii are determined theoretically and numerically. We give also explicit periodic solutions of the problem.

PACS number(s):

I. INTRODUCTION

The problem of the motion, of a rigid body around a fixed point, in the field of constant gravity, is one of the oldest in mechanics [1,2,3].

The problem can be formulated in terms of a time-independent Hamiltonian with three degrees of freedom (the Euler angles. In addition to the energy constant, there is present another constant of motion: the angular momentum component along the vertical. Rotation around the vertical is a cyclic coordinate in the Hamiltonian, which produces the new constant and allows the reduction of the problem to only two degrees of freedom.

The equations of motion (Euler-Poisson equations) are given by:

$$\begin{cases} A\dot{\omega} = A\omega \wedge \omega + e \wedge r \\ \dot{e} = e \wedge \omega \end{cases} \qquad (1)$$

These equations contain six parameters: three eigenvalues A_1, A_2, A_3 of the inertia operator, and three coordinates r_1, r_2, r_3 of the center of mass.

$\vec{e} = {}^t(\gamma_1, \gamma_2, \gamma_3)$, Is the unit vertical vector.

$\vec{\omega} = {}^t(\omega_1, \omega_2, \omega_3)$, Is the angular momentum vector.

The Euler-Poisson equations are Hamiltonian on the four-dimensional invariant manifolds:

$$M_c = \left\{ \left(\vec{\omega}, \vec{e} \right) \in \Re^6 : \langle A\vec{\omega}, \vec{e} \rangle = c \; ; \; \langle \vec{e}, \vec{e} \rangle = 1 \right\}$$

Since the problem reduces to only two degrees of freedom, it suffices to have one more independent integral in addition to the energy integral for complete integrability.

We list the known integrability cases:

1- Euler's case (1750): the fixed point is located at the center of mass ($r_1 = r_2 = r_3 = 0$) and rotation occurs freely without the influence of the torque. The new integral $\langle A\vec{\omega}, A\vec{\omega}$ is the square of the length of the angular momentum vector.

2- Lagrange's case (1788): the rigid body is asymmetric top ($A_1 = A_2$), with the center of mass on the axis of the symmetry $(r_1 = r_2 = 0)$ .The additional integral ω_3 is the projection of the angular momentum on the axis of dynamic symmetry.

3- Kovalevskaya's case (1889): in which the ellipsoid of inertia about the fixed point is symmetric, with different principal moment of inertia, one half the value of the other two ($A_1 = A_2 = 2A_3$) and the center of mass in the plane of equal moments of inertia ($r_3 = 0$) . The first integral discovered by Kovalevskaya is:

$$\left(\omega_1^2 - \omega_2^2 - \nu \vec{e_1} \right)^2 + \left(2\omega_1\omega_2 - \nu e_2 \right)^2$$

where

$$\nu = r/A_3 \; et \; r^2 = r_1^2 + r_2^2.$$

4-Goryatchev – Tchaplygin's case (G-T) (1900): In this particular case, the ellipsoid of inertia is symmetric having different moment equal to $\frac{1}{4}$ of the other inertia moments ($A_1 = A_2 = 4A_3$) , the center of mass on the symmetry plane ($r_3 = 0$) and the angular momentum component in the vertical direction equal to zero ($c = \langle A\vec{\omega}, e = 0$). The constant of motion is:

$$f = \omega_3 \left(\omega_1^2 + \omega_2^2 \right) - \nu \, \omega_1 e_3$$

In this paper we study the problem of rigid body in the last case (GT top). Especially, this paper is devoted to the investigation of the common level sets of the first integrals in order to describe the phase space topology of GT top. This study is given in a simple way in contrast to Smale's program for topological analysis of mechanical systems [4]. It is also completed by numerical illustrations.

II. TOPOLOGICAL ANALYSIS

The equations corresponding to the GT top become more samples in the special canonical variables: $(\;, \varphi_2, \varphi_3; L, I_2, I_3)$ called Andoyer-Deprit variables [5].

The Hamiltonian of GT top in the new variables $(\ , \varphi_2, \varphi_3; L, I_2, I_3)$ is given by:

$$H = \frac{I_2^2 + 3L^2}{8A_3} + r\left(\frac{L}{I_2}\sin l\ \cos\varphi_2 + \cos l\ \sin\varphi_2\right) \qquad (2)$$

The Hamilton-Jacobi equation separates in these variables. To see this, consider the canonical transformation to the variables (x, y, p_x, p_y):

$$\begin{cases} L = p_x + p_y; & l = \dfrac{x+y}{2} \\ I_2 = p_x - p_y & \varphi_2 = \dfrac{x-y}{2} \end{cases} \qquad (3)$$

The Hamiltonian function becomes:

$$H = \frac{{p_x}^3 - {p_y}^3}{2A_3(p_x - p_y)} + r\left(\frac{p_x}{p_x - p_y}\sin x + \frac{p_y}{p_x - p_y}\sin y\right) \qquad (4)$$

and the corresponding equations of motions:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} = \frac{2p_x + p_y}{2A_3} - \frac{r\,p_y}{(p_x - p_y)^2}(\sin x + \sin y) \\ \dot{y} &= \frac{\partial H}{\partial p_y} = \frac{2p_y + p_x}{2A_3} + \frac{r\,p_x}{(p_x - p_y)^2}(\sin x + \sin y) \\ \dot{p}_x &= \frac{-\partial H}{\partial x} = \frac{r p_x}{p_x - p_y}\cos x \\ \dot{p}_y &= \frac{-\partial H}{\partial y} = \frac{r\,p_y}{p_x - p_y}\cos y \end{aligned} \qquad (5)$$

Equating expression (4) to h and multiplying by (p_x - p_y), we see that it separates:

$$\frac{p_x^3}{2A_3} + r\ p_x\sin x - h\ p_x = \frac{p_y^3}{2A_3} - r\ p_y\sin y - h\ p_y$$

We put:

$$\begin{aligned} &\frac{p_x^3}{2A_3} + r\ p_x\sin x - h\ p_x = \Gamma \\ &\frac{p_y^3}{2A_3} - r\ p_y\sin y - h\ p_y = \Gamma \end{aligned} \qquad (6)$$

Function Γ is a first integral of the equations of motion (5). In the traditional variables (ω , e) it has the form:

$\Gamma = 2A_3^2\ f$ Where f is the constant of motion given by GT (see .I.)

From (4) and (5) taking account of (6), we obtain the following system of differential equations:

$$\begin{cases} \dot{p}_x = \pm\dfrac{\sqrt{\phi(p_x)}}{p_x - p_y} \\ \dot{p}_y = \pm\dfrac{\sqrt{\phi(p_y)}}{p_x - p_y} \end{cases} \qquad (7)$$

Where ϕ (z) is a polynomial of degree 6:

$$\phi(z) = r^2\ z^2 - \left(\Gamma + h\,z - \frac{z^3}{2A_3}\right)^2 .$$

The solution of these equations (7) can be expressed through hyperelliptic functions of time on the complexified manifolds:

$$A_{\mathbb{C}} = \left\{(x, y, p_x, p_y) \in \mathbb{C}^4 : \Gamma = cte, \mathrm{H} = cte\right\} \subset \mathbb{C}^4$$

Which is the complexified common level sets of the first integrals H and Γ of the GT top.

I.1- Topology of regular level sets:

In this section we shall give a detailed description of the real phase space topology i.e. the topology of the real level sets:

$$A_{\mathbb{C}} = \left\{(x, y, p_x, p_y) \in \Re^4 : \Gamma = cte, \mathrm{H} = cte\right\} \subset \Re^4$$

For doing that, we find first the bifurcation diagram B i.e. the set of the critical values of the first integrals H and Γ (like in Hénon-Heiles [6], Kolossoff [7]). The level set B is exactly the discriminant locus of the polynomial ϕ (z) defined in (7) can be written in the form:

$$\phi(z) = f_1(z)\quad f_2(z)$$

Where

$$f_1(z) = \frac{1}{2A_3}\left(z^3 + 2A_3\ (r-h)\ z - 2\,\mathrm{A}_3\ \Gamma\right)$$

$$f_2(z) = \frac{1}{2A_3}\left(-z^3 + 2A_3\ (r+h)\ z + 2\,\mathrm{A}_3\ \Gamma\right)$$

Thus the bifurcation diagram B is: B = $B_1 \cup B_2$

Where B_1 and B_2 are the bifurcation diagrams of the polynomials f_1 (z) and f2 (z) respectively. [8]

$$B_1 = \left\{(x, y, p_x, p_y) \in R^4 : A_3 = c_1, r = c_2\ ; \text{discriminant}\ (f_1(z)) = 0\right\} \subset \mathrm{R}^2$$

$$B_1 = \left\{(x, y, p_x, p_y) \in R^4 : A_3 = c_1, r = c_2\ ; \text{discriminant}\ (f_2(z)) = 0\right\} \subset \mathrm{R}^2$$

The polynomial

$$f_1(z) = \frac{1}{2A_3}\left(z^3 + 2A_3\ (r-h)\ z - 2\,\mathrm{A}_3\ \Gamma\right)$$

its discriminant D_1 is: $D_1 = Q^3 + R^2$

Where: $Q = \frac{2}{3}A_3(r-h),\ R = A_3\ \Gamma$

$f_1(z)$ has double roots if: D1 = 0 which corresponds to the case where the integrals H and Γ are not independent. Then, we find:

$$B_1 = \left\{\begin{array}{l} (x, y, p_x, p_y) \in R^4 : A_3 = c_1, r = c_2; \\ \Gamma = \pm\sqrt{\dfrac{8}{27}A_3}\,(h-r)^{3/2}, \Gamma = 0\ \ avec\ \ h > r \end{array}\right\}$$

We obtain the set B_2 in a similar way:

$$B_2=\left\{\begin{array}{l}(x,y,p_x,p_y)\in R^4 : A_3=c_1, r=c_2; \\ \Gamma=\pm\sqrt{\frac{8}{27}A_3}\,(h+r)^{3/2}, \Gamma=0 \quad avec \;\; h>-r\end{array}\right\}$$

The topological type of the real level set A_R may change only as the point (H,Γ) passes through B. The set $R^2\setminus B$ consists of 5 components (see fig.1.). Thus in each connected component of the set $R^2\setminus B$ the level set A_R has the same topological type.

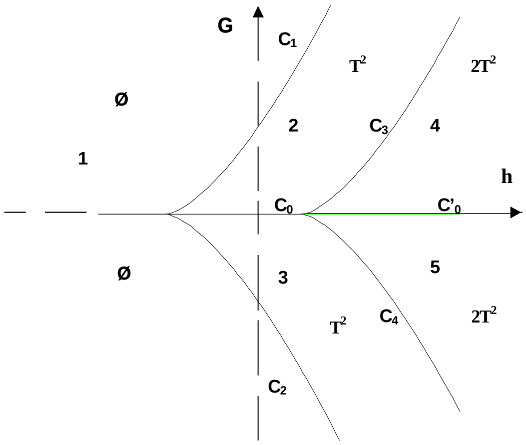


Figure.1- Diagram of bifurcation B

According to the classical Liouville theorem, for non-critical values of H and Γ, the level sets A_R is a finite union of two-dimensional Torii. Their number depends only upon the number and the location of the ovals (admissible intervals). To determine the ovals, it suffices to study the real roots of the polynomials f_1 (z) and $f_2(z)$ for different values of H and Γ (i.e. in each connected component of B). These roots are shown in table 1.

Dom	$f_1(z)$	$f_2(z)$	$\phi(z)$
1^+ (1) 1^-	$z_1>0$ $z_1<0$	$z_1'>0$ $z_1'<0$	$z_1'>z_1>0$ $z_1'<z_1<0$
(2)	z1 < 0	$z_3'<z_2'<0<z_1'$	$z_3'<z_2'<0<$ $z_1<z_1'$
(3)	z1 < 0	$z_2'<0<z_3'<z_1'$	$z_2'<z_1<0<$ $z_3'<z_1'$
(4)	$z_3<z_2<0<z_1$	$z_3'<z_2'<0<z_1'$	$z_3'<z_3<z_2<$ $z_2'<0<z_1<z_1'$
(5)	$z_2<0<z_3<z_1$	$z_2'<0<z_3'<z_1'$	$z_2'<z_2<0<$ $z_3<z_1<z_1'$

Table.1 -Location of roots of the polynomials $f_1(z)$ and $f_2(z)$ on domains delimited by diagram B.

z_i (i=1,2,3) are the roots of polynomial f_1 (z) and z_i' are those of the polynomial $f_2(z)$. Domains 1^+ and 1^- correspond to domain (1) for Γ 0 and Γ 0 respectively.

Using the formulas (5) and (7) and the condition that x, y, p_x, p_y are real, we obtain that A_R is an empty set only if (H,Γ) belongs to domain 1, on domains (2) to (5), A_R is a torus or a disjoint union of two Torii. That is obtained by the product of the "admissible " intervals for the variables p_x and p_y (see table.2.)

Dom	$p_x-plane:\Delta_1$	$p_y-plane:\Delta_2$	$\Delta_1\times\Delta_2$
1^+ 1^-	$[z_1,z_1']$ ϕ	ϕ $[z_2',z_1]$	ϕ ϕ
(2)	$[z_1,z_1']$	$[z_3',z_2']$	T^2
(3)	$[z_3',z_1']$	$[z_2',z_1]$	T^2
(4)	$[z_1,z_1']$	$[z_3',z_3]\cup[z_2,z_2']$	$2T^2$
(5)	$[z_3',z_3]\cup[z_1,z_1']$	$[z_2',z_2]$	$2T^2$

Table .2 - Admissible ovals and topological type of A_R.

I.2- Topology of singular level sets:

In section I.1, we found the topological type of the level set A_R far from the bifurcation diagram. Suppose now that the constants (H,Γ) passes through the bifurcation diagram B, then the topological type of A_R may change and bifurcation of Liouville take place.
In order to describe these bifurcations, it suffices to study the bifurcations of the real roots of the polynomials. Namely, each bifurcation of Liouville Torii is related to bifurcation of admissible intervals of the variables p_x and p_y .We find that there exist 6 types of possible bifurcations in the case of GT top.

The first one from is domain (2) to domain (1) passing through C_1. Denote this sequence by:

$$2 \longrightarrow C_1 \longrightarrow 1$$

The topological type of A_R in domain (2) is a two-dimensional torus T^2. On the curve C_1, this torus T^2 is contracted to the axial circle S^1 and then vanishes.
Denote this bifurcation as:

$$T^2 \longrightarrow S^1 \longrightarrow \phi$$

To prove that, it suffices to look at the bifurcations of roots of the polynomial $\phi(z)$, and more specifically the bifurcations of the admissible intervals Δ_1 and Δ_2. We have

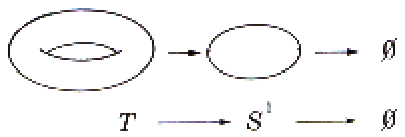
$$T^2 \longrightarrow S^1 \longrightarrow \phi$$

Remark: this result can be found by making use the Fomenko theory on bifurcations of Liouville Torii [9].

The others five bifurcations are proved in a similar way. Namely:

$$\begin{array}{ccccc} 3 & \longrightarrow & C_2 & \longrightarrow & 1 \\ T^2 & \longrightarrow & S^1 & \longrightarrow & \phi \end{array}$$

$$[z_3', z_1'] \times [z_2', z_1] \longrightarrow \{\bullet\} \times [z_2', z_1] \longrightarrow \phi \times [z_2', z_1]$$



$$\begin{array}{ccccc} 2 & \longrightarrow & C_3 & \longrightarrow & 4 \\ T^2 & \longrightarrow & S^1 \times (S^1 \vee S^1) & \longrightarrow & 2T^2 \end{array}$$

$$[z_1, z_1'] \times [z_3', z_2'] \longrightarrow [z_1, z_1'] \times [z_3', z_3 = z_2] \cup [z_3 = z_2, z_2'] \longrightarrow [z_3', z_3] \cup [z_2, z_2']$$

where $S^1 \vee S^1$ is a union of two circles having exactly one common point (figure eight) .

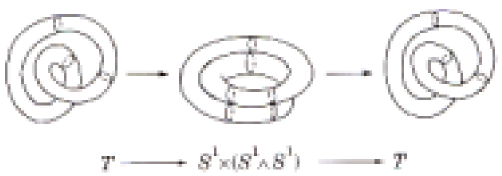
$$\begin{array}{ccccc} 3 & \longrightarrow & C_4 & \longrightarrow & 5 \\ T^2 & \longrightarrow & S^1 \times (S^1 \vee S^1) & \longrightarrow & 2T^2 \end{array}$$

$$[z_3', z_1'] \times [z_2', z_1] \longrightarrow [z_2', z_3] \times [z_3', z_2 = z_1] \cup [z_2 = z_1, z_1'] \longrightarrow$$

$$[z_3', z_3] \cup [z_1, z_1'] \times [z_2', z_2]$$

$$\begin{array}{ccccc} 2 & \longrightarrow & C_0 & \longrightarrow & 3 \\ T^2 & \longrightarrow & S^1 \times (S^1 \wedge S^1) & \longrightarrow & T^2 \end{array}$$

$$[z_1, z_1'] \times [z_3', z_2'] \longrightarrow [z_2' = z_1 = 0, z_1'] \times [z_3, z_2' = z_1 = 0] \longrightarrow [z_3', z_1'] \times [z_2', z_1]$$



$$\begin{array}{ccccc} 4 & \longrightarrow & C_0' & \longrightarrow & 5 \\ 2T^2 & \longrightarrow & T^2 \cup S^1 & \longrightarrow & 2T^2 \end{array}$$

$$[z_3', z_3] \cup [z_2, z_2'] \times [z_1, z_1'] \longrightarrow [z_3', z_3] \cup \{0\} \times [z_1, z_1'] \longrightarrow [z_3', z_3] \cup [z_1, z_1'] \times [z_2', z_2]$$

III – Periodic Solutions:

When the bifurcation of Liouville Torii takes place, the level set A_R becomes degenerate. Then we can have exceptional families of periodic solutions. We are interested by the cases where A_R contains a unique isolated circle which is a periodic solution. This is arises on the diagram of bifurcation B, more specifically when the first integrals H and Γ belong to the smooth curves C_1 and C_2 (see table .3.).

Cures	admissible ovals		topological type of $A_{\Re}$
	Δ_1	Δ_2	
C_1	$[z_1, z_1']$	$\{z_3' = z_2'\}$	S^1
C_2	$\{z_3' = z_1'\}$	$[z_2', z_1]$	S^1
C_3	$[z_1, z_1']$	$[z_3', z_3 = z_2] \cup [z_3 = z_2, z_2']$	$S' x(S' \vee S')$
C_4	$[z_3', z_2 = z_1] \cup [z_2 = z_1, z_1']$	$[z_2', z_3]$	$S' x(S' \vee S')$
C_0	$[z_2' = z_1 = 0, z_1']$	$[z_3, z_2' = z_1 = 0]$	$S' \times (S' \wedge S')$
C_0'	$[z_3', z_3] \cup \{0\}$	$[z_1, z_1']$	$T^2 \cup S^1$

Table .3. - Topological type of A_R ***for*** $(H, \Gamma) \in$ ***B***

III.1- Periodic Solution on C_1:

Consider a fixed periodic solution belonging to the curve C_1. It is parameterized by the equations (7) where p_x takes values in the admissible interval $[z_1, z'_1]$ and $p_y = z'_3 = z'_2 = \alpha$ is equal to the root of the polynomial $\phi(z)$ (see table .3). The values of the first integrals H and Γ are related by:

$$\Gamma = \sqrt{\frac{8}{27}}\, A_3 \; (h + r)^{3/2}$$

Then we obtain from (7) the following parameterization of the periodic solution $p_x(t)$:

$$dt = \pm 2A_3 \frac{(p_x - \alpha) dp_x}{|p_x - \alpha| \sqrt{(\beta - p_x)(p_x - \gamma)(p_x^2 + \gamma p_x + 2A_3(r - h) + \gamma^2)}}$$

where

$$\beta = z'_1 \;, \;\; \gamma = z_1 \quad (p_x \in [z_1, z'_1] \qquad p_x \in [\gamma, \beta]\;)$$

Since $\alpha < \gamma < \beta$ (see table .1.), by solving the Jacobi inversion problem:

$$\int_{\gamma}^{p_x} dt = \pm 2A_3 \int_{\gamma}^{p_x} \frac{dp_x}{\sqrt{(\beta - p_x)(p_x - \gamma)((p_x - m)^2 + n^2)}}$$

$$t = \pm 2A_3 \frac{1}{\sqrt{pq}} F(2arctg(\sqrt{\frac{q(\beta - p_x)}{p(p_x - \gamma)}}), \frac{1}{2}\sqrt{\frac{-(p-q)^2 + (\beta - \gamma)^2}{pq}})$$

where F (x, k) is the incomplete elliptic integral of the first kind ,

$$\begin{cases} p^2 = \beta^2 + \gamma\beta + \gamma^2 + 2A_3(r-h) \\ q^2 = 3\gamma^2 + 2A_3(r-h) \end{cases}$$

We obtain the periodic solution $p_x(t)$ and his period T is given by:

$$\oint dt = 2\int_{\gamma}^{\beta} \frac{(p_x - \alpha)}{\sqrt{\phi(p_x)}} dp_x$$

III.2- Periodic solution on C_2 :

In the same way , we determine also a fixed periodic solution belonging to the curve C_2 .In this case p_x is the double root of the polynomial $\phi(z)$: $p_x = z'_3 = z'_1 = \alpha_1 > 0$ and p_y takes values in the interval $p_y \in [z'_2, z_1]$. Denote $z'_2 = \beta_1$ and $z_1 = \gamma_1$,the periodic solution $p_y(t)$ is given by the inversion of :

$$t = \pm \frac{2A_3}{\sqrt{pq}} F(2arctg\sqrt{\frac{q(\gamma_1 - p_y)}{p(p_y - \beta_1)}}, \frac{1}{2}\sqrt{\frac{-(p-q)^2 + (\beta_1 - \gamma_1)^2}{pq}})$$

where

$$\begin{cases} p^2 = \beta_1^2 + \gamma_1\beta_1 + \gamma_1^2 + 2A_3(r+h) \\ q^2 = 3\gamma_1^2 + 2A_3(r+h) \end{cases}$$

and his period is:

$$T = \oint dt = 2\int_{\beta_1}^{\gamma_1} \frac{\alpha_1 - p_y}{\sqrt{\phi(p_y)}} dp_y$$

IV.- Numerical illustration :

By making use a method introduced by Poincaré and extended by Hénon [10]. The surfaces of section map shown in figures 2 to 10 give an illustration of the sequence of bifurcations of Liouville Torii.

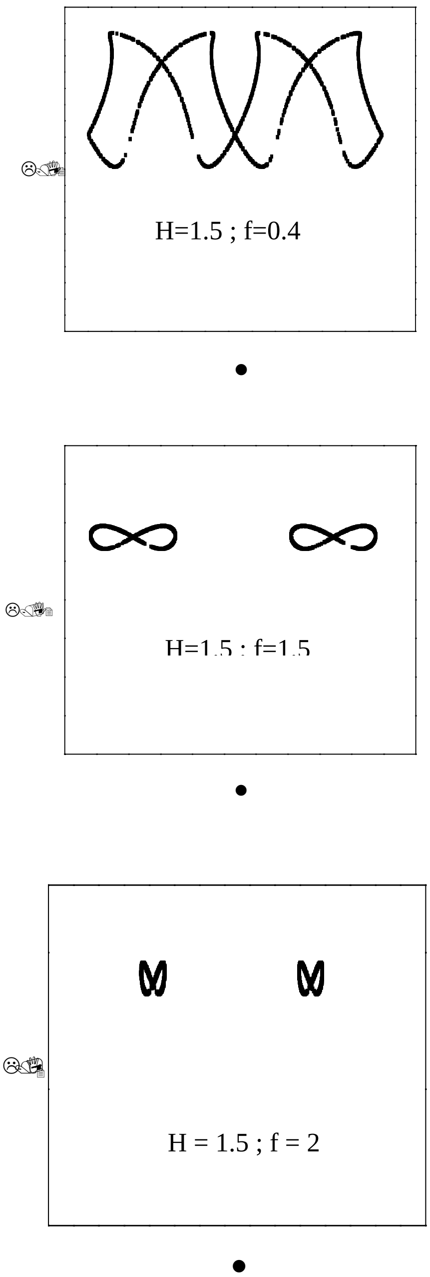


Fig.2.- Surfaces of section map in the plane (l, L/I_2); f > 0
$A_R \sim T^2$

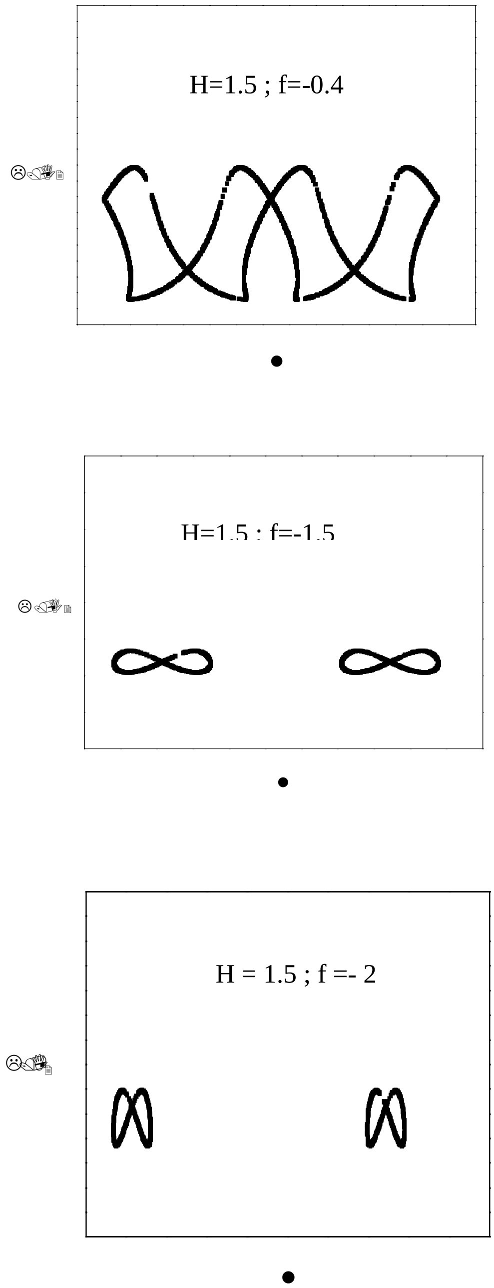
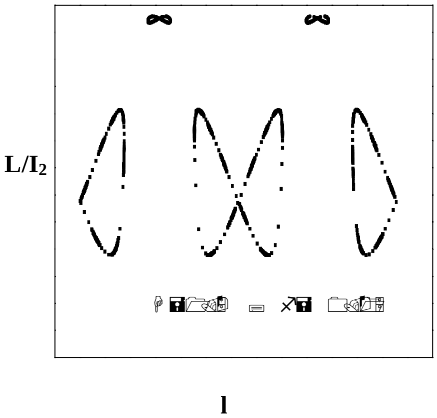


Fig.3.- Surfaces of section map in the plane (l, L/I_2); f < 0
$A_R \sim T^2$



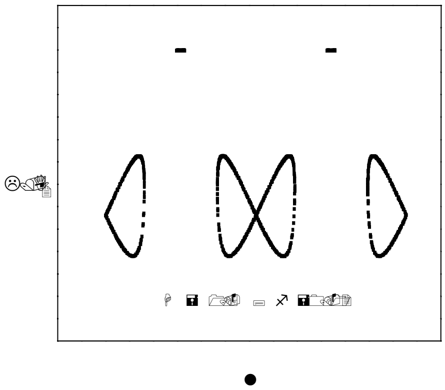


Fig.4.- Surfaces of section map in the plane (l, L/I_2); f > 0
A_R ~ $2T^2$

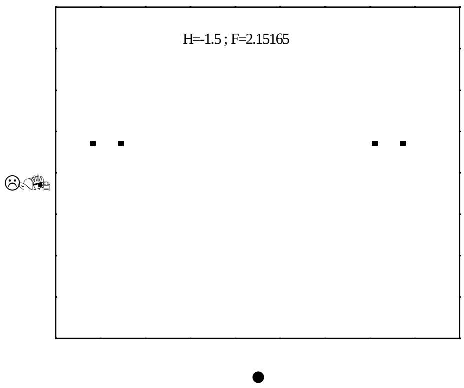
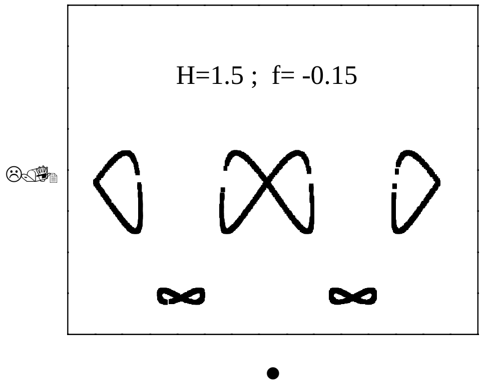


Fig.6.- Surface of section map in the plan (l, L/I_2) ; f > 0. **A_R ~ S^1**



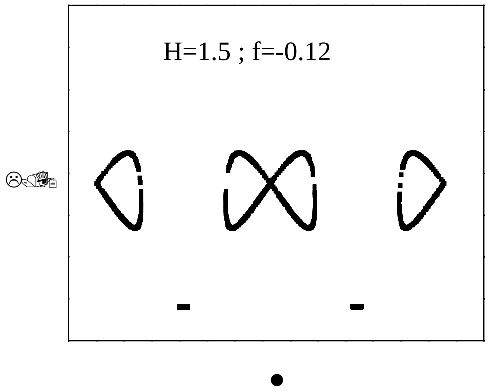


Fig.5.- Surfaces of section map in the plane (l, L/I_2); f < 0
A_R ~ $2T^2$

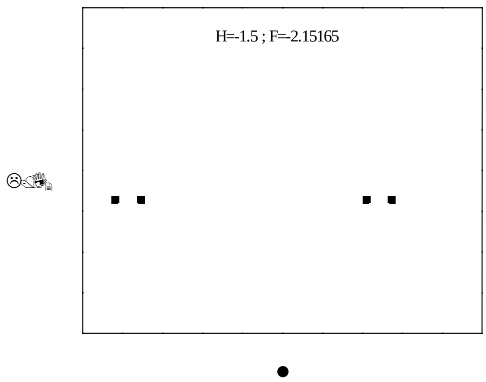


Fig.7.- Surface of section map in the plan (l, L/I_2) ; f < 0. **A_R ~ S^1**

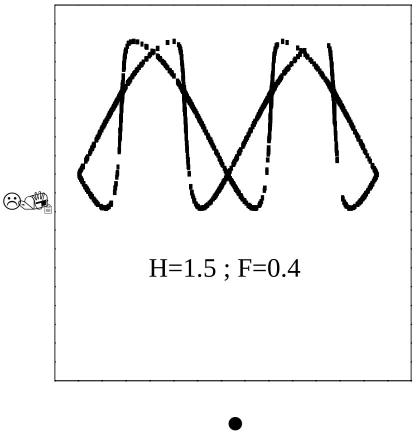


Fig.8- Surface of section map in the plan (l, L/I_2) ; f > 0
bifurcation $\mathbf{T^2} \longrightarrow \mathbf{2T^2}$

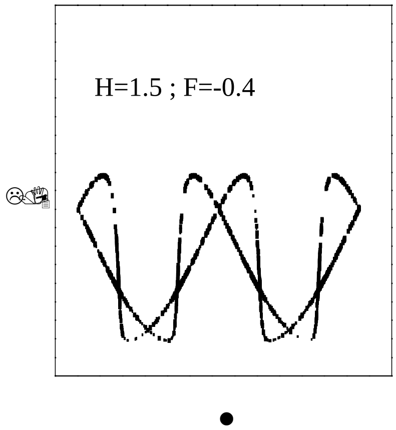


Fig.9- Surface of section map in the plan (l, L/I_2) ; f < 0
bifurcation $\mathbf{T^2} \longrightarrow \mathbf{2T^2}$

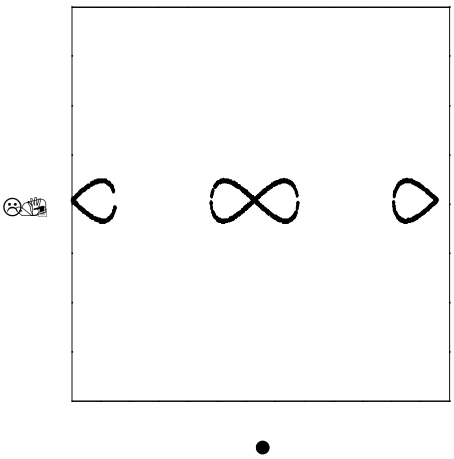


Fig.10- Surface of section map in the plan (l, L/I_2) ;
bifurcation $T^2 \longrightarrow T^2$

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