# Equation Governing the Thermal State of a Superconductor with Nonlinear Thermophysical Coefficients: Existence and Uniqueness

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**Abstract:** The aim of this paper is to prove the existence and uniqueness of the problem's solution:

$$(Pe)\begin{cases} \dfrac{c(u)\partial u}{\partial t} - \displaystyle\sum_{i=1}^{3} \frac{1}{\partial x_i}\left[k_{ij}(u)\frac{\partial u}{\partial x_i}\right] = \lambda F(u) + P(x,t) \quad on\, \Omega \times IR^+ \\ u \in E = C\left(IR^+, C^0(\Omega)\right) \cap C\left(IR^+, H_0^1(\Omega)\right) \cap C^1\left(IR^+, L^2(\Omega)\right) \\ \qquad u(x,t) = 0 \quad on \quad \partial\Omega \times IR^+ \\ \qquad u(x,0) = u_0 \quad for\ all\ x\ in\ \Omega \end{cases}$$

Our basic idea is to adapt an approach coupling between two processes:

1. An iterative scheme that helps to exploit all the results obtained in [11][15].

2. A construction of solutions sequence $(U^n)_{n \in IN}$ in $E$, which converges to the solution of the problem.

**Key Words:** Global existence, Non linear problem, Iterative processes, Uniqueness, Superconductivity.

## I. Framework study

Given a three dimensional superconductor Ω immersed in a cryogenic bath, it thermal state is managed by the problem $(Pe)$ [1][7][9][10][15]:

$$(Pe)\begin{cases} \dfrac{c(u)\partial u}{\partial t} - \displaystyle\sum_{i=1}^{3} \frac{1}{\partial x_i}\left[k_{ij}(u)\frac{\partial u}{\partial x_i}\right] = \lambda F(u) + P(x,t) \quad on\, \Omega \times IR^+ \\ u \in E = C\left(IR^+, C^0(\Omega)\right) \cap C\left(IR^+, H_0^1(\Omega)\right) \cap C^1\left(IR^+, L^2(\Omega)\right) \\ \qquad u(x,t) = 0 \quad on \quad \partial\Omega \times IR^+ \\ \qquad u(x,0) = u_0 \quad for\ all\ x\ in\ \Omega \end{cases}$$

Where $c$ and $k_{ij}$ are, respectively, the specific heat and components of the thermal conductivity tensor. In fact, the mathematical assumptions made about these thermophysical functions reflect their physical natures; they are defined on $IR^+$ to $IR$ of class $C^1$, strictly positive, lipschetz functions, and bounded with their derivatives.

λ is a bifurcation parameter assembling al intensive data of the problem, and $P$ is an input parasitic responsible of any possible thermal disturbance:

$$|P(x,t)| \le M \quad for\ all\ (x,t) \in \Omega \times IR^+$$

$F$ is the term of energetic competition between a power dissipated by Joule effect G, and that absorbed by the cryogenic bath Q:

$$F(u) = G(u) - Q(u)$$

The diversity of the techniques carrying out the superconductive state and\ maintaining its thermal stability, generates a whole of acceptable classes $U_{ad}^G$ for the term $F$. These classes are characterized primarily, but not only, by the existence of an element $u_1 > 1$ such\ as:

$$F(u_1) = 0$$

 

Let us note, moreover, that all elements of $U_{ad}^{G}$ are $C^2$ and null for the negative values $(u \leq 0)$. These assumptions represent the noted hypothesis $H_0$ In a quasi-general way, there are three techniques or systems of cooling ( Helium I, Helium II and the temperature control by the field edges). Each system is modeled by the hypothesis $H_0$ and an only hypothesis $H_m$ with $m \in \{1,2,3\}$ such as:

- $H_1$ :

there exists $u_2 > u_2$ such as $F(u_2) = 0$ *and* $F'(u) \leq 0$ *for all* $u \geq u_2$

- $H_2$ :

$$\lim_{u \to +\infty} F(u) = F_\infty$$

- $H_3$ :

$\exists \gamma_1 > 0$ *such as* $\lim_{u \to +\infty}(\frac{F(u)}{u}) \leq \gamma_1$ *and* $\lim_{u \to +\infty} F(u) = +$

Then, we define classes of this term as follows:

$$U_{ad}^{m} = \{ F \text{ checking } H0 \text{ and } Hm \} \quad \text{with } 1 \leq m \leq 3$$

Thus, the general class $U_{ad}^{G}$ is defined by:

$$U_{ad}^{G} = \bigcup_{m=1}^{3} U_{ad}^{m}$$

The whole of these assumptions are deduced from nature of experimental curves, given by the various cooling systems.

**Lemma 0.0.1** For all $m \in \{1,2,3\}$, $F$ checks the following growth [10][15] condition:

There exists two constants $F_m > 0$ *and* $q_m \geq 0$ such as:

$$|F(u)| \leq F_m \; |u|^{q_m} \tag{1}$$

In other words:

$$|F(u)| \leq a(x) + b \; |u|^{p/q} \tag{2}$$

Where $a$ is a function in $L^p(\Omega)$ and $p$ and $q$ are conjugates.

## II. Sequence of similar problems (*Pe, n*)

Let $u_s$ be the unique solution, in $E$ of the following problem $(Pe_{(x,t)})$ [11] :

$$\begin{cases} c(x,t)\dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{3} \frac{1}{\partial x_i}\left[k_{ij}(x,t)\frac{\partial u}{\partial x_i}\right] = \lambda F(u) + P(x,t) & on\ \Omega \times IR^+ \\ u(x,t) = 0 & on\ \partial\Omega \times IR^+ \\ u(x,0) = u_0 & for\ all\ x\ in\ \Omega \end{cases}$$

We set:

$$U^0(x,t) = u_s(x,t)$$

and we choose by convent:

$$c^0(x,t) = 1 \ \ and \ \ k_{ij}^{0}(x,t) = 1$$

Then, we define two sequences from $\Omega \times IR^+$ *to* $IR^+$ as following:

$$c^n(x,t) = c\left(U^{n-1}(x,t)\right) \ and \ \ k_{ij}^{n}(x,t) = k_{ij}\left(U^{n-1}(x,t)\right)$$

Where $U^{n-1}(x,t)$ is the only solution of the problem $P(e,n)$similar to $Pe$ defined by:

$$(Pe,n)\begin{cases} c^{n-1}(x,t)\dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{3} \frac{1}{\partial x_i}\left[k_{ij}^{n-1}(x,t)\frac{\partial u}{\partial x_i}\right] = \lambda F(u) + P(x,t) & on\ \Omega \times IR^+ \\ u(x,t) = 0 & on\ \partial\Omega \times IR^+ \\ u(x,0) = u_0 & for\ all\ x\ in\ \Omega \end{cases}$$

**Lemma 0.0.2** a) The function $c^n$ is $C^1$ on $\Omega \times IR^+$, strictly positive and bounded with all its partial derivatives.

b) For all $i,j \in \{1,2,3\}$, $k_{ij}^{n}$ is strictly positive, $C^1$ on $\Omega \times IR^+$ and bounded with all its partial derivatives.

**Proof.** As $U^{n-1}$ is a solution, there is a constant $\vartheta > 0$such as:

$$\sup_{t>0}\|U^{n-1}(x,.)\| \leq \vartheta < \infty \qquad for\ x\ on\ \Omega$$

This result allows to recover, for $c^n$, all mathematical properties associated to c. In other words, $c^n$ is $C^1$ on $\Omega \times IR^+$ strictly positive and bounded with all its partial derivatives; more precisely:

$$c_0 < c^n < c_1 \qquad (3)$$

The same reasons allow to conclude that $k_{ij}^n$ are strictly positive, increasing of class $C^1$ on $\Omega \times IR^+$, bounded with all its partial derivatives such as:

$$k_1 < k_{ij}^n < k_2 \qquad (4)$$

**Remark 0.0.3** The positive terms of the sequence $(U^n)_{n\geq 0}$ are well defined, because that for every$n \geq 1$, $U^n$ exists and is unique in $E$.

## III. Convergence of the sequence $(U^n)_{n\geq 0}$

For each iteration $n$, we define two functions:

$$F_{c^n}(u) = \frac{F(u)}{c^{n+1}(x,t)}, \quad P_{c^n}(x,t) = \frac{P(x,t)}{c^{n+1}(x,t)}$$

**Lemma 0.0.4** 1. Let $m \in \{1,2,3\}$ we have:

$$F \in U_{ad}^m \quad \Leftrightarrow \quad F_{c^n} \in U_{ad}^m$$

2. The function $P_{c^n}$ is bounded for every $n > 0$.

**Proof.** For all $n \geq 0$, we have:

$$c^n(x,t) > 0 \ \ for\ all\ (x,t)\ \ on\ \ \Omega \times IR^+$$

$F_{c^n}$ checks for all $v \in E$:

$$F_{c^n}(u) \leq \frac{1}{c_0} F(u)$$

Continuity and differentiability of $F_{c^n}$ are deducted from those of $F$ and $c$. This proves the first point of Lemma.

For $P_{c^n}$, we have:

$$|P_{c^n}(x,t) = | \leq \frac{1}{c_0} |P(x,t)| \quad for\ all\ (x,t)\ \ on\ \ \Omega \times IR^+$$

Hence the second result.

**Proposition 0.0.5** For all $n \geq 1$, there is a continuous linear operator from $D_{L_t} = L^2(IR^+, H_0^1(\Omega)) \cap C^0(IR^+, L^2(\Omega))$ to $L^2(IR^+, H^{-1}(\Omega))$ such as the problem $(Pe,n)$can be written as the followings semi-linear form :

$$\frac{\partial u}{\partial t} + L_{t,n}(u) = \lambda F_{c^n}(u) + P_{c^n} \qquad (6)$$

(6)

And

$$L_{t,n}(u) = \sum_{i,j=1}^N l_n^1(x,t)\frac{\partial u}{\partial x_i} - \sum_{i,j=1}^N l_n^2(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} \qquad (7)$$

Where $l_n^1$ and $l_n^2$ are two explicit functions.

**Proof.** Since $c^n$ is strictly positive for all $n \geq 1$ we put:

$$\tilde{e}_{ij}^n = \frac{k_{ij}^n}{c^{n+1}}$$

As $\tilde{e}_{ij}^n \in C^1(\Omega \times IR^+)$, we can define $e_{ij}^n$ by:

$$e_{ij}^n = \frac{\partial}{\partial x_i}\left(\frac{k_{ij}^n}{c^{n+1}}\right) - \frac{1}{c^{n+1}}\left(\frac{\partial k_{ij}^n}{\partial x_i}\right)$$

If we develop the term:

$$\sum_{i,j=1}^N \frac{1}{\partial x_j}\left[k_{ij}^{n-1}(x,t)\ \frac{\partial u}{\partial x_j}\right]$$

and then we introduce the linear operator $L_{t,n}$ from $D_{L_t}$ into $L^2(IR^+, H^{-1}(\Omega))$ :

$$L_{t,n}(u) = \sum_{i,j=1}^N (e_{ij}^n - \tilde{e}_{ij}^n)\frac{\partial u}{\partial x_i} - \sum_{i,j=1}^N \tilde{e}_{ij}^n \frac{\partial^2 u}{\partial x_i \partial x_j} \qquad (8)$$

For all $n \geq 1$, $(Pe,n)$ is transformed in the following semi-linear form:

$$\frac{\partial u}{\partial t} + L_{t,n}(u) = \lambda F_{c^n}(u) + P_{c^n}$$
(9)

With:

$$l_n^1 = e_{ij}^n - \tilde{e}_{ij}^n \qquad and \qquad l_n^2 = \tilde{e}_{ij}^n$$

Let us show now that the operator $L_{t,n}$ is continuous.

Indeed, using the same techniques given in [11][15], $v \in H_0^1(\Omega)$, we have:

$$|< L_{t,n}(u), v > | \leq M\lambda_1 \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)}$$

Where $M$ is the following positive constant:

$$\mathrm{M} = \sup_{i,j=1} |e_{ij}^n| \mathrm{N}^2 + \sup_{i,j=1} |\tilde{e}_{ij}^n| \mathrm{N}^2 + 2\mathrm{CN}^2$$

**Lemma 0.0.6.** There are strictly positive constants $m_1, m_2$ *and* $\beta$ such as:

For all
$i,j \in \{1,2,3\}$, *for all* $n, m$ *and for all* $(x,t)$ *on* $\Omega \times IR^+$
we have the two following majorations:

$$|\tilde{e}_{ij}^n(x,t) - \tilde{e}_{ij}^m(x,t)| \leq m_1 |U^n(x,t) - U^m(x,t)| \quad (10)$$

$$|e_{ij}^n(x,t) - e_{ij}^m(x,t)| \leq m_2 |U^n(x,t) - U^m(x,t)| + \beta \quad (11)$$

**Proof.** Let $(x,t) \in \Omega \times IR^+$. Since $k_{ij}^n$ are bounded we have:

$$|\tilde{e}_{ij}^n(x,t) - \tilde{e}_{ij}^m(x,t)| \leq \frac{k_2}{c_1^2} |c^{n+1}(x,t) - c^{m+1}(x,t)|$$

As $c$ is lipschtz function, we obtain:

$$|\tilde{e}_{ij}^n(x,t) - \tilde{e}_{ij}^m(x,t)| \leq \frac{k_2 l_2}{c_1^2} |U^n(x,t) - U^m(x,t)|$$

Hence the first point.

Now, to prove the second point, since the derivatives of $k_{ij}^n$ and $c$ are bounded:

$$|e_{ij}^n(x,t) - e_{ij}^m(x,t)| \leq \frac{\tilde{k}_2}{c_1^2} |U^n(x,t) - U^m(x,t)| + \beta$$

$\beta$ is a constant independent of the difference $|U^n(x,t) - U^m(x,t)|$.

**Remark 0.0.7** The operator $L_{t,0}$ is defined by:

$$L_{t,0}(u) = \sum_{i,j=1}^N \left( \frac{\partial}{\partial x_i}\left(\frac{1}{c(x,t)}\right) - \frac{1}{c(x,t)} \right) \frac{\partial u}{\partial x_i} - \sum_{i,j=1}^N \frac{1}{c(x,t)} \frac{\partial^2 u}{\partial x_i \partial x_j}$$
(12)

**Theorem 0.0.8** For all $2 \leq p < \infty$ the sequence $(\mathrm{U}^n)_{n \geq 0}$ converges in the space $L^p(\Omega)$.

**Proof.** To prove the theorem, we have to prove that $(\mathrm{U}^n)_{n \geq 0}$ is a Cauchy sequence in $L^p(\Omega)$ for all $p \in [2, +\infty[$.

Let $n, m \geq 1$ since the terms $\mathrm{U}^n$ and $\mathrm{U}^m$ are solutions, respectively, of $(Pe, n+1)$ and $(Pe, m+1)$, we obtain:

$$\frac{\partial \mathrm{U}^n}{\partial t} + L_{t,n}(\mathrm{U}^n) = \lambda F_{c^n}(\mathrm{U}^n) + P_{c^n}(x,t)$$

$$\frac{\partial \mathrm{U}^m}{\partial t} + L_{t,m}(\mathrm{U}^m) = \lambda F_{c^m}(\mathrm{U}^m) + P_{c^m}(x,t)$$

In $]0, t^*[$ small set of time, we can write:

$$\mathrm{U}^n - \mathrm{U}^m = \int_0^{t^*} (L_{t,n}(\mathrm{U}^m) - L_{t,n}(\mathrm{U}^n))dt + \lambda \int_0^{t^*} (F_{c^n}(\mathrm{U}^n) - F_{c^m}(\mathrm{U}^m))dt$$

$$+ \int_0^{t^*} (P_{c^n}(x,t) - P_{c^m}(x,t))dt$$

Or we have for all functions f and g the following inequality:

$$|f(x)+g(x)|^p \le (|f(x)|+|g(x)|)^p \le 2^p(|f(x)|^p+|g(x)|^p) \quad (13)$$

Now, integrating over Ω:

$$\int |U^n - U^m|^p dx \le 2^{2p}\int\int_0^{t^*} |L_{t,n}(U^m)-L_{t,n}(U^n)|^p\, dtdx + 2^{2p}\lambda^p \int \int_0^{t^*} |F_{c^n}(U^n)-F_{c^m}(U^m)|^p\, dtdx + 2^p \int\int_0^{t^*} |P_{c^n}(x,t)-P_{c^m}(x,t)|^p\, dtdx = \sum_{i=1}^{3} I_i$$

Now, we proceed by treating each term of the above increase.

- For the quantity $I_1$, we have:

$$|P_{c^n}(x,t)-P_{c^m}(x,t)| \le P(x,t)\left(\frac{c^{m+1}(x,t)-c^{n+1}(x,t)}{c^{n+1}(x,t)c^{m+1}(x,t)}\right) \quad (14)$$

Or $P$ is bounded, so:

$$I_3 \le \frac{2^{2p} P_\infty^p l_2^p}{c_0^{2p}} t^* \|U^n - U^m\|^p_{L^p(\Omega)}$$
(15)

- Similarly for $I_2$ we have:

$$|F_{c^n}(U^n)-F_{c^m}(U^m)| \le |F_{c^n}(U^n)-F_{c^n}(U^m)| + |F_{c^n}(U^m)-F_{c^m}(U^m)| \quad (16)$$

Or $F$ checks the growth condition, so using the second form of lemma 0.0.1:

$$|F_{c^n}(U^n)-F_{c^n}(U^m)| \le \frac{1}{c_0}a(x) + \frac{b}{c_0}|U^n - U^m|^{\frac{p}{q}}$$

Then

$$\int |F_{c^n}(U^n)-F_{c^n}(U^m)|^p\, dx \le \frac{1}{c_0^p}\int |a(x)|^p\, dx + \frac{b^p}{c_0^p}\int |U^n - U^m|^q\, dx$$

As $q=\frac{p}{p-1}>1$, we have:

$$\int |F_{c^n}(U^n)-F_{c^n}(U^m)|^p\, dx \le \frac{1}{c_0^p}\int |a(x)|^p\, dx + \frac{b^p}{c_0^p}\int |U^n - U^m|^p\, dx$$

If we use, on one hand, the first expression of the growth condition for $F(U^m)$ and on the other hand the fact that $c$ is a lipschitz function, we find:

$$|F_{c^n}(U^m)-F_{c^m}(U^m)| \le \frac{l_2 F_\infty}{c_0^2}|U^m|^{q^*}|U^n - U^m|$$

Finally, we obtain the following increasing:

$$I_2 \le \frac{2^{2p}\lambda^p}{c_0^p}\|a\|^p_{L^p(\Omega)} t^* + \frac{2^{2p}\lambda^p}{c_0^p} t^* \|U^n - U^m\|^p_{L^p(\Omega)} + \frac{2^{2p}\lambda^p F_\infty^p l_2^p}{c_0^{2p}} \left(\|U^m\|_{L^\infty(\Omega)}\right)^{pq^*} t^* \|U^n - U^m\|^p_{L^p(\Omega)}$$

- For the first term $I_1$, we have:

$$|L_{t,m}(U^m)-L_{t,n}(U^n)| \le |L_{t,m}(U^m)-L_{t,m}(U^n)| + |L_{t,m}(U^n)-L_{t,n}(U^n)|$$

To establish the increases to follow, we use the fact that derivatives of the functions $U^0, U^{01}, \ldots, U^n$ (element of $E$) are bounded. In other words, for all $n \ge 1$ there are

two constants $M_1^n$ and $M_2^n$ such as:

$$\left|\frac{\partial U^n}{\partial x_i}\right| \le M_1^n \quad and \quad \left|\frac{\partial^2 U^n}{\partial x_i \partial x_j}\right| \le M_2^n$$

Then we take:

$$M_1 = \sup_n M_1^n \quad and \quad M_2 = \sup_n M_2^n$$

We note that the linear operator $L_{t,m}$ checks:

$$|L_{t,m}(U^m)-L_{t,m}(U^n)| \le 2M_1 N^2\left(\sup_{i,j=1,2,\ldots,N}|e_{ij}^m| + \sup_{i,j=1,2,\ldots,N}|\tilde{e}_{ij}^m|\right) + 2M_1 N^2 \sup_{i,j=1,2,\ldots,N}|\tilde{e}_{ij}^m|$$

And using lemma 0.0.6:

$$|L_{t,m}(U^n)-L_{t,n}(U^n)| \le \sum_{i,j=1}^{N}|e_{ij}^m - e_{ij}^n|\left|\frac{\partial U^n}{\partial x_i}\right|$$

$$+\sum_{i,j=1}^{N}\left|\tilde{e}_{ij}^{m}-\tilde{e}_{ij}^{n}\right|\left|\frac{\partial U^n}{\partial x_i}\right|+\sum_{i,j=1}^{N}\left|\tilde{e}_{ij}^{m}-\tilde{e}_{ij}^{n}\right|\left|\frac{\partial^2 U^n}{\partial x_i \partial x_j}\right|$$

Then, we find that:

$$\left|L_{t,m}(U^m)-L_{t,n}(U^n)\right| \leq N^2(m_1+m_1)M_1|U^n-U^m|+N^2\beta M_1+N^2 m_1 M_2|U^n-U^m|$$

By integrating this expression over $\Omega$, we have:

$$I_1 \leq 2^{2p}\, N^{2p}(m_1+m_1)^p M_1{}^p t^* \|U^n - U^m\|_{L^p(\Omega)}^p$$

$$+$$

$$2^{2p}\, N^{2p}(m_1)^p M_2{}^p t^* \|U^n - U^m\|_{L^p(\Omega)}^p + |\Omega|\gamma t^*$$

With $\gamma$ is the constant given by:

$$\gamma = 4^{2p}\left(N^{2p}\beta^p M_1{}^p + \left(\sup_{i,j=1,2,\dots,N}\left|e_{ij}^{m}\right| + \sup_{i,j=1,2,\dots,N}\left|\tilde{e}_{ij}^{m}\right|\right) M_1{}^p + 2^p N^{2p} \sup_{i,j=1,2,\dots,N}\left|\tilde{e}_{ij}^{m}\right|^p M_2{}^p\right)$$

So finally we have the following for all $I_i$, $i = 1,2,3$:

$$\left(\frac{1-\alpha t^*}{t^*}\right)\|U^n - U^m\|_{L^p(\Omega)}^p \leq \rho$$

(17)

Where $\alpha$ is a constant strictly positive and does not depend on the difference( $U^n - U^m$), and $\rho$ the constant :

$$\rho = \left(\frac{2^{2p}\lambda^p}{c_0^p}\|a\|_{L^\infty(\Omega)} + |\Omega|\gamma\right)$$

Inspired by the technique used by Cazenave and Haraux, quite to choose $t^* \in ]0,\frac{\varepsilon}{\rho+\alpha\varepsilon}[$ , , for all $\varepsilon$ small enough, $(U^n)_{n\geq 0}$ is a Cauchy sequence in $L^p(\Omega)$ for all $p \in [2,+\infty[$. Then, there is a limit $\varphi$ in $L^p(\Omega)$ for $p \in [2,+\infty[$. such as:

$$U^n \to U \text{ in } L^p(\Omega) \;\; for\ n \to \infty$$

## IV. Convergence of the similar problems sequence

We have shown that the sequence of solutions $(U^n)_{n\geq 0}$ converges to a unique solution in $L^p(\Omega)$ for all $p \in [2,+\infty[$. In the following, we will show that this limit is the global solution to the problem $(Pe)$

**Lemma 0.0.9** 1. There is a subsequence $(c^{n_i})_n$ of $(c^n)_n$ which converges uniformly to a limit $c^\infty$ in $IR^+$

2. For all $i,j$ we can extract a subsequence of $(k_{ij}^n)_n$ which converges to a continuous function $k_{ij}^\infty$ in $IR^+$.

**Proof.** By virtue of theorem 0.0.8, there is $\varphi$ in $L^p(\Omega)$ limit of $(U^n)_{n\geq 0}$ . Then, the partial converse of the Lebesgue dominated convergence theorem prove that there is a subsequence

$$U^{n_i}(x,t) \to U(x,t) \quad for \quad all \quad (x,t) \;\in\; \Omega \times IR^+$$

In other words, for all $\varepsilon > 0$, there is an integer $N_1$ such as for all $n_i \geq N_1$ and for all $(x,t)$.

$$|U^{n_i}(x,t) - U(x,t)| < \epsilon \tag{18}$$

Since $c$ and $k_{ij}$ are continuous, we have:

$$c(U^{n_i}(x,t)) \to c\big(U(x,t)\big) = c^\infty$$

$$k_{ij}(U^{n_i}(x,t)) \to k_{ij}\big(U(x,t)\big) = k_{ij}{}^\infty$$

Now, we will prove the uniform convergence of $(c^{n_i})_n$ to $c^\infty$ In fact, we have:

$$|c^{n_i}(x,t) - c^\infty(x,t)| = |c(U^{n_i}) - c(U)|$$

Although, $c$ is a lipschitz function:

$$|c^{n_i}(x,t) - c^\infty(x,t)| \leq l_2|U^{n_i}(x,t) - U(x,t)|$$

Then, we obtain:

$$|c^{n_i}(x,t) - c^\infty(x,t)| \leq \delta = l_2\varepsilon$$

hence the result.

Similarly, since $k_{ij}$ are lipschitz functions, we find that convergence is uniform. This proves that the limit is continuous.

**Corollary0.0.10** 1. The functions sequence $(\tilde{e}_{ij})$ (resp. $(e_{ij})$) contains a subsequence which converges uniformly to a limit function $\tilde{e}_{ij}^{\infty}$( resp. $e_{ij}^{\infty}$).

2. The functions sequence $F_{c^n}$ contains a subsequence $F_{c^{n_i}}$ which converges uniformly to a function $F_{c^\infty}$.

3. There exists a subsequence of $P_{c^n}$ which converges to a function $P_{c^\infty}$.

Now, we denote $L(D_{L_t}, L^2(IR^+, H^{-1}(\Omega))$ the space of the continuous operators from $D_{L_t}$ to $H^{-1}(\Omega)$ with the norm:

$$\|T\|_{L(D_{L_t}, L^2(IR^+, H^{-1}(\Omega))} = \|T\|_L = \sup_{\|v\|_{H_0^1} \le 1} \|Tv\|_{H_0^1}$$

And we define the operator $L_{t,\infty}$ by:

$$L_{t,\infty}(u) = \sum_{i,j=1}^{N} (e_{ij}^{\infty} - \tilde{e}_{ij}^{\infty}) \frac{\partial u}{\partial x_i} - \sum_{i,j=1}^{N} \tilde{e}_{ij}^{\infty} \frac{\partial^2 u}{\partial x_i \partial x_j}$$

**Proposition0. 0. 11** The sequence of operators $(L_{t,n_i})$ converges, in $L(D_{L_t}, L^2(IR^+, H^{-1}(\Omega))$ to the operator $L_{t,\infty}$.

**Proof.** Let $v$ an element of the unit ball $B(0,1)$ of $H_0^1(\Omega)$ We have:

$$\|L_{t,n_i}(v) - L_{t,\infty}(v)\|_{H_0^1} = \|L_{t,n_i}(v) - L_{t,\infty}(v)\|_{L^2}$$

$$+ \sum_{k=1}^{N} \left\| \frac{\partial}{\partial x_k} (L_{t,n_i}(v) - L_{t,\infty}(v)) \right\|_{L^2}$$

For the first term, we obtain:

$$\int |L_{t,n_i}(v) - L_{t,\infty}(v)|^2 dx \le \sum_{i,j=1}^{N} \int \left|(e_{ij}^{n_i} - e_{ij}^{\infty})\right|^2 \left|\frac{\partial v}{\partial x_i}\right|^2 dx$$

$$+ \sum_{i,j=1}^{N} \int \left|(\tilde{e}_{ij}^{n_i} - \tilde{e}_{ij}^{\infty})\right|^2 \left|\frac{\partial v}{\partial x_i}\right|^2 dx$$

$$+ \sum_{i,j=1}^{N} \int \left|(\tilde{e}_{ij}^{n_i} - \tilde{e}_{ij}^{\infty})\right|^2 \left|\frac{\partial^2 v}{\partial x_j \partial x_i}\right|^2 dx$$

By passing to the limit, we find:

$$\int |L_{t,n_i}(v) - L_{t,\infty}(v)|^2 dx \le (\varepsilon_1^2 + \varepsilon_2^2) \|\nabla v\|_{L^2}^2 + \varepsilon_1^2 \sum_{i=1}^{N} \|\nabla V_1\|_{L^2}^2$$

With

$$V_1 = \sum_{j=1}^{N} \frac{\partial v}{\partial x_j}$$

Then, we obtain:

$$\int |L_{t,n_i}(v) - L_{t,\infty}(v)|^2 dx \le \varepsilon_3 \quad (19)$$

In addition, for the second term we have:

$$\left|\frac{\partial (e_{ij}^{n_i} - e_{ij}^{\infty})}{\partial x_k}\right| \le \eta_1$$

and

$$\left|\frac{\partial (\tilde{e}_{ij}^{n_i} - \tilde{e}_{ij}^{\infty})}{\partial x_k}\right| \le \eta_2$$

That gives:

$$\left\|\frac{\partial}{\partial x_k}\left(L_{t,n_i}(v) - L_{t,\infty}(v)\right)\right\|_{L^2} \le N(\eta_1^2 + \eta_2^2)\|\nabla v\|_{L^2}^2 + (\varepsilon_1^2 + \varepsilon_2^2 + \eta_2^2)\|\nabla V_1\|_{L^2}^2$$

$$+ \varepsilon_2^2 \|\nabla V_2\|_{L^2}^2$$

Where,

$$V_2 = \sum_{j=1}^{N} \frac{\partial v_k}{\partial x_j}$$

$$\sum_{k=1}^{N} \left\|\frac{\partial}{\partial x_k}\left(L_{t,n_i}(v) - L_{t,\infty}(v)\right)\right\|_{L^2} \le \eta_3 \quad \textbf{(20)}$$

This proves:

$$\sup_{\|v\|_{H_0^1}} \left\| L_{t,n_i}(v) - L_{t,\infty}(v) \right\|_{H_0^1} = \left\| L_{t,n_i} - L_{t,\infty} \right\|_L \leq \mu$$

**Theorem 0.0.12** The problem $(Pe)$ admits a unique solution in $E$.

**Proof .** We showed that the sequence of solutions $(U^n)_{n\geq 0}$ converges to a unique solution $\varphi$ in $L^p(\Omega)$ for all $p \in [2, +\infty[$.Furthermore, from the sequence of operators $(L_{t,n})$ ,we could extract a subsequence $(L_{t,n_i})$ which converges to a continuous linear operator $(L_{t,\infty})$ in $L(D_{L_t}, L^2(IR^+, H^{-1}(\Omega)))$.

We can deduce that the subsequence of transferred semi linear problems $(P(e, n_i))$ converges to the following problem $(P(e,\infty))$:

$$\frac{\partial u}{\partial t} + L_{t,\infty}(u) = \lambda F_{c^\infty}(u) + P_{c^\infty}$$

This semi linear problem begins in its original form as follows:

$$(P\varepsilon,\infty)\begin{cases} c^\infty(x,t)\dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{3}\frac{1}{\partial x_i}\left[k_{ij}^\infty(x,t)\frac{\partial u}{\partial x_i}\right] = \lambda F(u) + P(x,t) \quad on\,\Omega \times IR^+ \\ u(x,t) = 0 \quad on\ \partial\Omega \times IR^+ \\ u(x,0) = u_0 \quad for\ all\ x\ in\ \Omega \end{cases}$$

By passing to the operators limit:

$$(Pe,\infty)\begin{cases} c(u)\dfrac{\partial u}{\partial t} - \displaystyle\sum_{i=1}^{3}\frac{1}{\partial x_i}\left[k_{ij}(u)\frac{\partial u}{\partial x_i}\right] = \lambda F(u) + P(x,t) \quad on\,\Omega \times IR^+ \\ u(x,t) = 0 \quad on\ \partial\Omega \times IR^+ \\ u(x,0) = u_0 \quad for\ all\ x\ in\ \Omega \end{cases}$$

The problem $(Pe,\infty)$ which is the same as the problem $(Pe)$admits the unique solution $\varphi$ limit of $(U^n)_{n\geq 0}$ .

## V. Conclusion

In this paper, we studied the question of existence and uniqueness of the solution of the problem managing the thermal state of a superconductor in three dimensions. Give the triple nonlinearity of the problem, we adapted a strategy of complementarity with work given in [11]. By this approach, we could prove the existence and uniqueness of the solution of the of the tripling nonlinear problem, in a well with the appropriate physical and technological realities imposed by the superconductors.

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