Multicritical Behaviours in One-Dimensional Traffic Flow

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Abstract: The effect of the on-ramp and off-ramp positions i_1 and i_2, respectively, on the one dimensional-cellular automaton traffic flow behaviour, is investigated numerically. The on-ramp and off-ramp rates at i_1 and i_2 are α_0 and β_0, respectively. However, in the open boundary conditions, with injecting and extracting rates α and β and using parallel dynamics, several phases occur, depending on the position of $i_1$by respect to i_2. Namely, low density phase (LDP), intermediate density phase (IDP), plateau current phase (PCP) and high density phase (HDP). Furthermore, critical, tricritical and multicritical behaviours take place in the (i_1, α_0) phase diagrams.

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I. Introduction

The field of transport have attracted increased attention in recent years [1-4]. This interest is due primarily to the fact that transportation problems are related to the global behaviour of systems with many elements interacting at short distances, such as the vehicles travelling on the streets or information which travel over the internet network.

In particular, the investigation of open traffic systems with on- and off-ramps is quite popular at the moment [5-10]. One reason for this is the impact of the understanding of varying the different flow rates in order to optimise the total flow or trip times.

Among the different methods of investigation and simulation of highway traffic flow, asymmetric simple exclusion process (ASEP) is the most promising [11-13]. Indeed, ASEP is the simplest driven diffusive system where particles on a one-dimensional lattice hop with asymmetric rates under excluded volume constraints.

The effect of on- and off-ramps have been largely studied by several researchers in the traffic flow field [5,11]. Our principal aim in this paper is to bring an explanation on the following point: let us suppose that we have a highway running in an urban conglomeration and that there are an access from urban conglomeration to the highway and an exit from the highway. We want to understand where the access and exit positions must be localised in order to maximize the flux of cars in the road. In other words, we want to study the effect of the on-ramp and off-ramp positions on the one dimensional-cellular automaton traffic flow behaviour, in the open boundaries case. To this end, we have established the phase diagrams in the (i_1, α_0)-plane. Depending on the injecting and extracting rate values, an adequate localization of the on- and off-ramp positions leads to the appearance of new phases, topologies and several multicritical behaviours. Moreover, to compare our results to those where only one off-ramp was taken into account [5], quantitative differences can be understood from the behaviour of average density and phase diagrams for different parameters.

The paper is organised as follows: Model and method are given in section 2; section 3 is reserved to results and discussion; the conclusion is presented in section 4.

II. Model and method

We consider a one-dimensional lattice of length L. Each lattice site is either empty or occupied by one particle. Hence the state of the system is defined by a set of occupation numbers $\tau_1, \tau_2, ..., \tau_L$, while $\tau_i = 1$ = 1 ($\tau_i = 0$) means that the site i is occupied (empty). We suppose that the main road is single lane, an on-ramp and an off-ramp connect the main road only on single lattice i_1 for entry and on single lattice i_2 for way out. During each time interval Δt, each particle jump to the empty adjacent site on its right and does not move otherwise $(i \neq i_2)$. Δt is an interesting parameter that enables the possibility to interpolate between the cases of fully parallel $(\Delta t = 1)$ and random sequential $(\Delta t \to 0)$ updates [12]. Particles are injected, by a rate $\alpha\Delta t$, in the first site being to the left side of the road if this site is empty, and particles enter in the road by site i_1, with a probability $\alpha_0\Delta t$ without constraint, if this site is empty. While, the

particle being in the last site on the right can leave the road with a rate $\beta\Delta t$ and particles removed on the way out with a rate $\beta_0\Delta t$. At site i_1 (i_2) the occupation (absorption) priority is given to the particle which enter in the road (particle leaving the road). Hence the cars, which are added to the road, avoid any collision.

If the system has the configuration $\tau_1(t), \tau_2(t), \ldots, \tau_L(t)$ at time t it will change at time $t+\Delta t$ to the following:

For $i = i_1$,

$$\tau_i(t+\frac{\Delta t}{2}) = 1$$

with probability

$$q_i = \tau_i(t) + \left[\alpha_0(1-\tau_i(t)) - \tau_i(t)(1-\tau_{i+1}(t))\right]\Delta t$$

and

$$\tau_i(t+\frac{\Delta t}{2}) = 0$$

with probability

$1-q_i$. Where i_1 and α_0 denote the position of the entry site and the injection rate, respectively.

For $i = i_2$,

$$\tau_i(t+\frac{\Delta t}{2}) = 1$$

with probability

$$q_i = \tau_i(t) + \left[\tau_{i-1}(1-\tau_i) - \beta_0\tau_i(t)\right]\Delta t$$

and

$$\tau_i(t+\frac{\Delta t}{2}) = 0$$

with probability $1-q_i$. Where i_2 and β_0 denote the position of the absorbing site and the absorbing rate, respectively.

For $1 < i < L$ with $i \neq i_1$ and $i \neq i_2$,

$\tau_i(t+\Delta t) = 1$

with probability

$$q_i = \tau_i(t) + \left[\tau_{i-1}(t)(1-\tau_i(t)) - \tau_i(t)(1-\tau_{i+1}(t))\right]\Delta t$$

and

$$\tau_i(t+\Delta t) = 0$$

with probability $1-q_i$.

For $i = 1$,

$$\tau_1(t+\Delta t) = 1$$

with probability

$$q_1 = \tau_1(t) + \left[\alpha(1-\tau_1(t)) - \tau_1(t)(1-\tau_2(t))\right]\Delta t$$

and

$$\tau_1(t+\Delta t) = 0.$$

with probability $1-q_1$.

For $i = L$,

$$\tau_L(t+\Delta t) = 1$$

with probability

$$q_L = \tau_L(t) + \left[\tau_{L-1}(t)(1-\tau_L(t)) - \beta(1-\tau_L(t))\right]\Delta t$$

and

$$\tau_L(t+\Delta t) = 0$$

with probability $1-q_L$.

III. Results and Discussion

In our numerical calculations, the rule described above is updated in parallel, $\Delta t = 1$, i.e. during one update step the new particle position do not influence the rest and only the previous positions have to be taken into account [13]. During each of the time steps, each particle moves one site unless the adjacent site on its right is occupied by another particle.

In the following, the temporal average of any parameter is computed for 5.10^4 to 10^5 time steps. Starting the simulations from random configurations, the system reaches a stationary state after a sufficiently large number of time steps. In all our simulations, we averaged over 60-100 configurations.

As we have mentioned previously, our aim in this paper is to study the effect of the on- and off-ramps positions i_1 and i_2, respectively, for different values of α, β_0 and β, on the average density and flux in the chain. The study is made in the open boundary conditions case. The length of the road studied in this paper is L=1000. Moreover, our goal here is to study the cases where i_1 is upstream or downstream i_2, we then fixed $i_2 = 500$ and varied i_1.

The figure 1 gives the variation of the average density ρ as a function of the injecting rate α_0 for several values of the on-ramp position i_1 and for $\alpha = 0.1$, $\beta = 0.1$ and $\beta_0 = 0.4$.

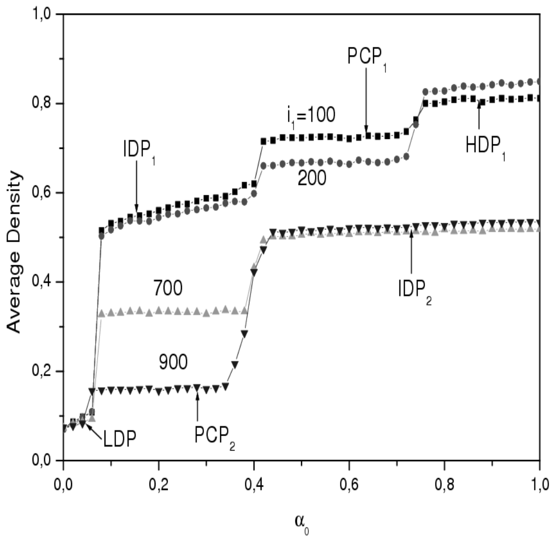

Figure 1: Average density ρ versus the injection rate α_0 for α=0.1, β=0.1, β_0=0.4. The number accompanying each curve denotes the value of i_1.

However, when the on-ramp is located upstream the off-ramp, the system studied exhibits four phases, namely: i) The low density phase (LDP), where the average density increases when increasing the rate of injected particles α_0. ii) The intermediate density phase (IDP_1) characterised by a smoothly increase of the density. iii) The plateau current phase (PCP_1) for which the density and current are constant in a special interval of α_0. iv) The high density phase (HDP_1) in which, for high values of α_0, the current decreases and the density reaches a large value and remains constant. On the other hand, when i_1 is located downstream i_2, the high density phase disappears. Note that we add index 1 to each phase when this one is specific to the case $i_1 < i_2$ and index 2 in the contrary case. In addition, when increasing i_2 for a given value of α_0, the average density remains constant in the LDP, while it decreases in the IDP_1 and PCP then increases in the HDP_1. Moreover, an inversion point is located at a special value of $\alpha_0 = \alpha_{0i}$ which corresponds to the PCP_1-HDP_1 transition. This inversion can be explained as follows:

- For $i_1 < i_2$: if $\alpha_0 < \alpha_{0i}$ and i_1 increases, the section between i_1 and i_2 decreases reducing the average density (α is small). While, if $\alpha_0 > \alpha_{0i}$, there is a fast accumulation of particles in the section between i_1 and L (β being weak) which increases the average density.

-For $i_1 > i_2$: if α_0 is small, the increase in i_1 reduces the zone $i_1 - L$ thus decreasing the average density. Whereas, when α_0 is large, the accumulation of particles between i_1 and L is compensated by the reduction in the section $i_1 - L$. The average density remains thus constant.

In addition, if $i_1 < i_2$, we note that the IDP, which doesn't appear in the model where only the off-ramp is taken into account [5], occurs for the intermediate values of α_0 ($\alpha_{0c1} < \alpha_0 < \alpha_{0c2}$). α_{0c1} and α_{0c2} correspond to the transition between $LDP - IDP_1$ and $IDP_1 - PCP_1$, respectively. While, the PCP_1 arises between two critical values α_{0c2} and α_{0c3} of injecting rate α_0, where α_{0c3} corresponds to the $PCP_1 - HDP_1$ transition, which coincide with α_{0i}. We note that this transition disappears when i_1 is located after i_2 (Figure 1).

In order to have a suitable criterion for determination of the nature of the transition, we identify the first order transition (abrupt transition) by a jump in the average density or by the existence of a peak in the derivative of $\rho(\alpha_0)$ with respect to α_0 [13]. This means that the transitions shown in figure 1 are of first order type. Collecting the results illustrated in figure 1, the different regions are given on the phase diagram (i_1, α_0) shown in Figure 2.

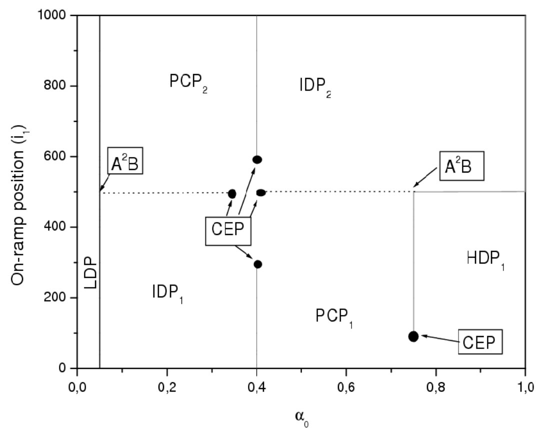

Figure 2: Phase diagram (i_1, α_0) for $\alpha = 0.1$, $\beta = 0.1$, $\beta_0 = 0.4$. The solide and dote lines indicate the first and second order transitions, respectively.

Beside this, such phase diagram exhibits the critical and multicritial behaviours. The critical end-points, around which there is no distinction between the phases, are denoted by CEP.

In order to describe the different entities in the phase diagrams, the Griffiths notations [14] will be adopted: the multicritical point $A^m B^n$ denotes the intersection of m lines of first order and n lines of second order. The A^3 indicates then the triple point, which is the intersection of three lines of first order. The critical point, which is the intersection of a line of first order and a line of second order, is denoted by C. However, the figure 2 shows two multicritical points A^2B and five single critical end-points CEP. Indeed, for a given value of α_0, the transitions $\mathrm{IDP}_1 - \mathrm{PCP}_2$ and $\mathrm{PCP}_1 - \mathrm{IDP}_2$ are of the second order, while the transition $\mathrm{HDP}_1 - \mathrm{IDP}_2$ is of the first order. We identify the second order transition by respect to the average density, which is continuous but presents an angular point [12].

For low values of α, β and β_0 ($\alpha = 0.1$, $\beta = 0.1$ and $\beta_0 = 0.1$), the (i_1, α_0) phase diagram is presented in figure 3, where all the transitions are of the first order, except that located between $\mathrm{PCP}_1 - \mathrm{IDP}_2$. Moreover, such figure exhibits three critical end-points and one multicritical point of type A^2B.

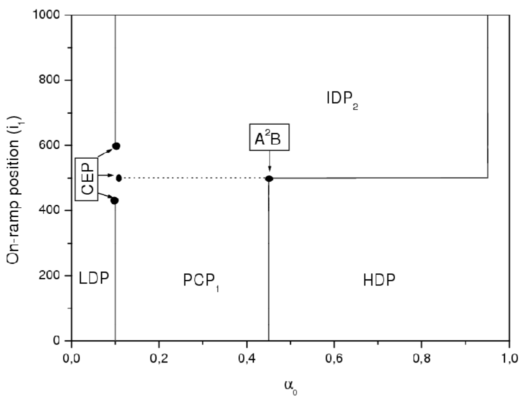

Figure 3: Phase diagram (i_1, α_0) for $\alpha = 0.1$, $\beta = 0.1$, $\beta_0 = 0.1$.

The comparison of figures 2 and 3 shows up the effect of β_0 on the (i_1, α_0) phase diagram. Indeed, for intermediate value of β_0, the IDP_1 and PCP_2 take place.

Furthermore, for a sufficiently large value of β_0 ($\beta_0 = 0.8$, as shown in figure 4), the IDP_1 disappears, two critical points (C), two triple points A^3 and two end-points (CEP) occur.

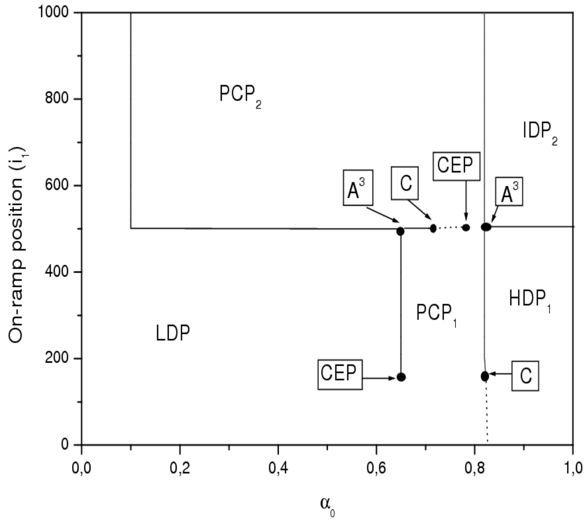

Figure 4: Phase diagram (i_1, α_0) for $\alpha = 0.1$, $\beta_0 = 0.8$ and $\beta = 0.1$.

Finally, to highlight the effect of β, we present on figure 5 the (i_1, α_0) phase diagram for $\alpha = 0.1$, $\beta_0 = 0.1$ and $\beta = 0.3$. In this case, the system exhibits one multicritical point A^2B, one critical point C, one triple point A^3 and two critical end-points CEP. From figures 2, 3 and 5, we deduce that the IDP_1 arises for intermediate values of extracting rates β or β_0.

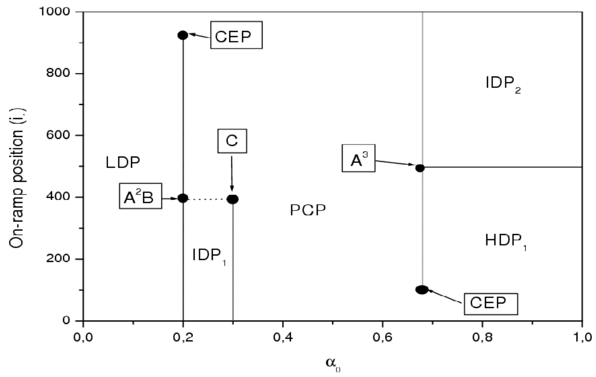

Figure 5: Phase diagram (i_1, α_0) for $\alpha = 0.1$, $\beta_0 = 0.1$ and $\beta = 0.3$.

IV. Conclusion

Using numerical simulations, we have studied the effect of the on- and off-ramp positions on the traffic flow behaviour of a one dimensional-cellular automaton, with parallel update. When i_1 is upstream i_2, an inversion point takes place in $\rho(\alpha_0)$, at $PCP_1 - HDP_1$ transition. Depending on the values of α, β and β_0, the (i_1, α_0) phase diagram exhibits different topologies. The transitions between different phases are of the first or second order. Furthermore, the system exhibits a multicritical, critical and critical end-points.

V. References

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