# Global Existence and Uniqueness of a Field Modelling the Thermal State of Three-dimensional Superconductor ($1\leq N \leq 3$)

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In this paper, we will study the global existence of a thermic field modelling the physical state of a three - dimensional superconductor. The superconductor is considered as a widely bounded regular field, with the usual physical data.
Key words: Nonlinear, existence, monotonous operator, supraconductors.

## I. Study framework

The thermic state of a superconductor field Ω, is given by the following mathematic model: [1], [5], [6], [7], [10], [11], [12], [13] and [15]:

$$c(x,t) = \frac{\partial u}{\partial t} - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j}[k_{i,j}(x,t)\frac{\partial x_i}{\partial t}] =$$

$$\lambda F(u) + P \quad in\,\Omega\, xIR^{+} \qquad (1)$$

$$u/_{\partial\Omega xIR^{+}} = 0 \qquad (2)$$

$$u/_{\partial\Omega x\{0\}} = u_0 \qquad (3)$$

With:
- c : Specific heat of material.
- $k_{ij}$ : Components of the thermic conductivity tensor.
-F : Non linearity term, representing the source of energy.
-P : Thermic perturbation .
-λ : Assembling parameter of all intensive amounts of the problem.

### 0.1.1 Functional framework

Functional framework, describing our study, is based on the mathematic reading of physics data. [12], [14], and [16]

For an initial data $u_0 \in C^0(\Omega)$, finding a solution u(t ; :) in the space

$$E = C(IR^{+}, C^{0}(\Omega)) \cap C(IR^{+}, H_0^1(\Omega)) \cap C^1(IR^{+}, L^2(\Omega))$$

From a physical point of view, the components of the tensor of thermic conductivity,for the spacial variable x, are continuous, strictly positive functions, and in a lower temperatures, strictly increasing. This, suppositions are mathematicly modelised by the following assumption $H_k$:

For all i and j:
-$k_{ij}$ is strictly positive and increasing of $C^1(\Omega x\ IR^{+})$. There are two constants independent of x and t $k_1$ and $k_2$ such as

$$k_1 < k_{i,j}(x,t) < k2 \quad for \quad all \quad x \in \Omega \quad and \quad t > 0$$

- For all 1 ≤ I ≤ N, the $k_{ij}$ are bounded; i-e there is M > 0 such as:

$$\left|\frac{\partial k_{i,j}}{\partial x_i}(x,t)\right| \leq M \quad for \quad all \quad x \in \Omega \quad and \quad t > 0$$

Specific heat c, is a continuous strictly positive function for the spacial variable x, checking $H_c$:
-c is $C^1(\Omega x\ IR^{+})$, strictly positive, and there are two constants independent of x and t $c_0$ and $c_1$ such as:

$$c_0 < k_{i,j}(x,t) < c_1 \quad for \quad all \quad x \in \Omega \quad and \quad t > 0$$

-There exists $c_2 > 0$ independent of x and t such as:

$$\left|\frac{\partial c}{\partial x_i}(x,t)\right| \leq c_2 \quad for \quad any \quad (x,t) \ \in \Omega x IR^{+}$$

F is the term of energetic competition between a power dissipated by Joule effect G, and that absorbed by the cryogenic bath Q:

F(u) = G(u) - Q(u)

The diversity of the techniques carrying out the

superconductive state and maintaining its thermal stability, generates a whole of acceptable classes $U_{ad}^G$ for the term F. These classes are characterized primarily, but not only, by the existence of an element $u_1 > 0$ such as:

$F(u_1) = 0$

Let us note, moreover, that all elements of $U_{ad}^G$ are $C^2(E)$ and null for the negative values ($u \cdot 0$). These assumptions represent the noted hypothesis $H_0$. In a quasi-general way, there are three techniques or systems of cooling ( Helium I, Helium II and the temperature control by the field edges). Each system is modeled by the hypothesis $H_0$ and only hypothesis $H_m$ with $m \in \{1; 2; 3\}$ such as:

-$H_1$:

$$there\ exists\ u_2 > u_1\ such\ as\ F(u_2) = 0\ and$$

$$F'(u) \leq 0\ for\ all\ u \geq u_2$$

-$H_2$:

$$\lim_{u\to\infty} F(u) = F_\infty$$

-$H_3$:

$$\exists \gamma_1 > 0\ such\ as\ \lim_{u\to\infty}\left(\frac{F(u)}{u}\right) < \gamma_1\ and$$

$$\lim_{u\to\infty} F(u) = +\infty$$

Then, we define classes of this term as follows:

$$U_{ad}^m = \{F\ checking\ H_0\ \ H_m\}\ with\ 1 \leq m \leq 3$$

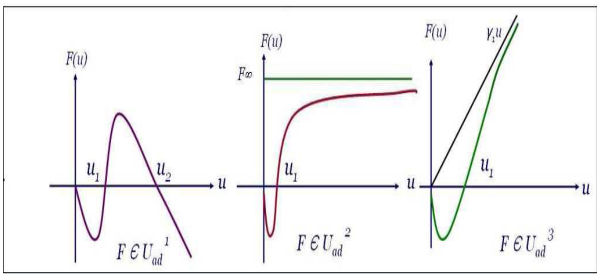
Thus, the general class $U_{ad}^G$ is defined by:

$$U_{ad}^G = \bigcup_{m=1}^{3} U_{ad}^m$$

The whole of these assumptions is deduced from nature of experimental curves, given by the various cooling systems.

**Remark 0.1.1** Let us recall that the physical data are obtained for example in [12], [14], and [15]

-$U_{ad}^1$ characterizes cooling system by Helium II .
-$U_{ad}^2$ gives cooling model by extrimities.
-$U_{ad}^3$ characterizes cooling system based on Helium I.




### 0.1.1 Thermic model

For c and $k_{ij}$ verifyning the hypothesis $H_c$ and $H_k$, and for F element of $U_{ad}^G$, the superconductor thermic state is a ¯eld u, element of E checking:

$$Pe\begin{cases} c(x,t)\dfrac{\partial u}{\partial t} - \sum\limits_{i,j=1}^{N} \dfrac{\partial}{\partial x_j}[k_{i,j}(x,t)\dfrac{\partial x_i}{\partial t}] = \lambda F(u) + P\ in\Omega xIR^+ \\ u/_{\partial\Omega xIR^+} = 0 \\ u/_{\partial\Omega x\{0\}} = u_0 \end{cases}$$

Balance states are solutions of the corresponding elliptic nonlinear problem:

$$\begin{cases} -\sum\limits_{i,j=1}^{N} \dfrac{\partial}{\partial x_j}[k_{i,j}(x,0)\dfrac{\partial u}{\partial x_i}] = \lambda F(u) + P\ \ in\Omega xIR^+ \\ u/_{\partial\Omega} = 0 \end{cases}$$

## 0.2 Simply nonlinear case

We remind, here, some results concerning the case of the constant thermophysic coe±cients; In fact this case represents the isotropic state. Without loss of generality, we suppose that:

$c = 1$ and that $\overline{\overline{k}} = Id$

The problem becomes:

$$(Pe,1,Id)\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u = \lambda F(u) + P\ in\Omega xIR^+ \\ u/_{\partial\Omega xIR^+} = 0 \\ u/_{\partial\Omega x\{0\}} = u_0 \end{cases}$$

Lemma **0.2.1** For all F element of UG ad, we have :

a) There exists two constants α and β > 0 such as:

$$uF(u) \leq \alpha u^2\ \ for\ all\ |u| \geq \beta$$

b) There exists two constants α > 0 and μ > 0 such as:

$$F(u) \leq \mu|u|\ \ for\ all\ |u| < a$$

Proof:

a) For $F \in U_{ad}^2$ , $F(u) \leq 0$ for all $u \geq u_2$

When F is in $U_{ad}^1$ , F behaves as as constant beside $+\infty$. So there exists $u_*$ such as:

$$For\ u \geq u_*\ F(u) \leq F_\infty$$

We have the lemma for:

$$\beta = \sup(u_*, u_2) \ \ and \ \ \alpha = F_\infty$$

For $U_{ad}^0$ , there exists $u_0$:

$$For \ \ u \geq u_0, \ \ uF(u) \leq (F_0 + 1)u^2$$

Is then : $\alpha = F_0 + 1$ and $\beta = u_0$

Thus, we can choose for all F in $U_{ad}^G$ :

$$\beta = \sup(u_*, u_2, u_0), \ \ \alpha = \sup(u_2, F_\infty, F_0 + 1)$$

b) is trivial. It's enough to choose: $a = u_1$ *and* $\mu > 0$.

Theorem **0.2.2** For all (F; $u_0$) $\in$ $U_{ad}^G$ x$C_0(\Omega)$, the problem (Pe;1;Id) admits a unique global solution.

The conditions required by the class $U_{ad}^G$ (continuity of F, local Lipschtizty and F(0) = 0) allow to apply the theorem of chapter 5rd of [3]. Which shows the local existence of the solution. Lemma 0:3:1 gives the results of the global existence.

**Proposition 0.2.3** If μ is a constant checking:

Then, there exists a constant A < ∞, such as if

$\|u_0\| C^0(\Omega) \cap H_0^1(\Omega) < u_1 A$

That:

a) The global solution verifies:

$\|u_0\| C^0(\Omega) \cap H_0^1(\Omega) \leq A \|u_0\| C^*(\Omega) \cap H_0^1(\Omega) e^{-(\lambda-\mu)t}$

$for \ \ all \ \ t \geq 0$

b) If F $\in$ $U_{ad}^2$ , then:

$$\|u(.,t)\| C^0(\Omega) \cap H_0^1(\Omega) \leq e^{-\lambda t}$$

This result flows from the lemma [0:3:2], the proposition 5:3:7 of [3] and the theorem [0:3:3].

### 0.3 Nonlinear case: c and $k_{ij}$ variable functions

We remind the aim of this paper is to prove, the global existence of solution of the problem. This will be done in two stages: study without term of nonlinearity, and then into the general framework. The basis idea is the ability to construct an operator $L_t$, which reponds to the suitable proprieties to apply the results of [5] [8] and [9].

#### 0.3.1 Operator depending of time

Because of the speci¯c heat is non null, the equation (1) can be put as:

$$\frac{\partial u}{\partial t} - \frac{1}{c(x,t)} \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_j}[k_{i,j}(x,t)\frac{\partial u}{\partial x_i}] = \frac{\lambda F(u)}{c(x,t)} + \frac{P}{c(x,t)}(4)$$

If we will note by: $\widetilde{c}$ the inverse of $c(x; t)$, we have the new quantities:

$$F_c(u) = \widetilde{c}F(u) \quad (5)$$

$$P_c = \widetilde{c}P \quad (6)$$

$$D_{ij} = \widetilde{c}k_{ij} \quad (7)$$

$$\widetilde{D}_{ij} = \widetilde{c}\frac{\partial k_{ij}}{\partial x_i} \quad (8)$$

($Pe$) will be given by:

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{N} \widetilde{D}_{ij}\frac{\partial u}{\partial x_i} - \sum_{i,j=1}^{N} D_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} =$$

$$\lambda F_e(u) + P_e \ \ in \ \ \Omega x IR^+ \quad (9)$$

$$u/_{\partial\Omega x IR^+} = 0 \quad (10)$$

$$u/_{\partial\Omega x\{0\}} = u_0 \quad (11)$$

If we introduce the linear operator $L_t$:

$$L_t(u) = \sum_{i,j=1}^{N}(e_{ij} - \widetilde{e}_{ij})\frac{\partial u}{\partial x_i} - \sum_{i,j=1}^{N}\widetilde{e}_{ij}\frac{\partial^2 u}{\partial x_i \partial xj} \quad (12)$$

With:

$$\widetilde{e}_{ij} = D_{ij} \quad (13)$$

$$e_{ij} = \frac{\partial D_{ij}}{\partial xi} - \widetilde{D}_{ij} \quad (14)$$

The problem ( $\widetilde{\text{Pe}}$ ) is put in the following new form:

$$\frac{\partial u}{\partial t} + L_t(u) = \lambda F_c(u) + P_c \quad (15)$$

We note the spaces:

$$D_{L_t} = L^2(IR^+, H_0^1(\Omega)) \cap C^0(IR^+, L^2(\Omega))$$

and dual space

$$H^{-1}(\Omega) = (H_0^1(\Omega))'$$

**Lemma 0.3.1** $\widetilde{e}_{ij}$ *and* $e_{ij}$ *are elements of*

$L^\infty(IR^+, L^\infty(\Omega))$

**Proof :** As the field Ω is bounded, and according to assumptions *Hc* and *Hk*, we have: for all $(x; t) \in \Omega$x$IR^+$

$$\left|\tilde{e}_{ij}(x,t)\right| \le \frac{k_2}{c_0} \quad and \quad \left|e_{ij}(x,t)\right| \le \frac{2M}{c_0} + \frac{k_2 c_2}{c_0^2} \quad (16)$$

**Proposition 0.3.2** *Lt is continuous from $D_{Lt}$ into*

$$L^2(IR^+, H^{-1}(\Omega))$$

Proof:

Let v $\in H_0^1(\Omega)$ from (12) and (16) we have :

$$\left|< L_t(u), v > \le \sum_{i,j=1}^{N} \int_\Omega \left| e_{ij} \left\| \frac{\partial v}{\partial x_i} \right\| v \right| dx \right| +$$
$$\sum_{i,j=1}^{N} \int_\Omega \left| \tilde{e}_{ij} \left\| \frac{\partial v}{\partial x_i} \right\| v \right| dx + \sum_{i,j=1}^{N} \int_\Omega \left| \tilde{e}_{ij} \frac{\partial^2 v}{\partial x_i \partial xj} v \right| dx \le$$
$$N^2 \sup_{i,j=1}^{N} \left| e_{ij} \|\Delta u\| L^2(\Omega) \|v\| L^2(\Omega) \right| +$$
$$N^2 \sup_{i,j=1}^{N} \left| \tilde{e}_{ij} \|\Delta u\| L^2(\Omega) \|v\| L^2(\Omega) \right| +$$
$$\sum_{i,j=1}^{N} \int_\Omega \left| \tilde{e}_{ij} \frac{\partial^2 v}{\partial x_i \partial xj} v \right| dx \qquad (17)$$

If we put: $\tilde{v}_{ij} = e_{ij} v$, then $\tilde{v}_{ij} \in L^2(\Omega)$ and

$\partial_{lk} \tilde{v}_{ik} = (\partial_{lk} \tilde{e}_{ik}) v + \tilde{e}_{ik} (\partial_{lk} v)$

The hypothesis *Hk and Hc* givess that $\tilde{v}_{ij}$ is an element of $D_{Lt}$. On the one hand,

intergration by parts gives us:

$$\sum_{i,j=1}^{N} \int_\Omega \left| \tilde{v}_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} \right| dx \le N \left\| \Delta \tilde{v}_{ij} \right\| L^2(\Omega) \|\Delta u\| L^2(\Omega) \qquad (18)$$

On the other hand:

$$\left\| \Delta \tilde{v}_{ij} \right\| L^2(\Omega) \le 2CN \|\Delta v\| L^2(\Omega) \|v\| L^2(\Omega) \quad (19)$$

Which allows to deduce from (17); (18); and (19):

$$\left| < L_t(u), v > \right| \le B \|\Delta u\| L^2(\Omega) \|v\| L^2(\Omega)$$

With

$$B = \sup_{i,j=1}^{N} \left| e_{ij} \right| N^2 + \sup_{i,j=1}^{N} \left| \tilde{e}_{ij} \right| N^2 + 2CN^2$$

Using the Poincarre inequality, we can conclude that:

$$B = N^2 [\sup_{i,j=1}^{N} \left| e_{ij} \right| + \sup_{i,j=1}^{N} \left| \tilde{e}_{ij} \right| + 2C]$$

**Remark 0.3.3** Constant:

$$B = \frac{2N^2}{c_0} [M + \frac{k_2 c_2}{2c_0} + \frac{G}{c_0}] \quad (20)$$

*is maximum. Where M is the constant related to the thermaic conductivity (Assumption $H_k$), and $c_0$ and $c_1$ those given by Hc.*

**0.3.2 Case** where $F_c(u) = 0$

We devote this part for the study of the problem:

$$(P_{e,L,N}) \begin{cases} \frac{\partial u}{\partial t} + L_t(u) = P_c(x,t) \\ u /_{\partial \Omega x IR^+} = 0 \qquad in \quad \Omega x IR \\ u /_{\partial \Omega x \{0\}} = u_0 \end{cases}$$

Let $a0$ a bilinear form de¯ned by:

$$a_0(t,u,v) = \sum_{i,j=1}^{N} \int_\Omega \tilde{e}_{ij} \frac{\partial u \partial v}{\partial x_i \partial xj} dx + \sum_{i,j=1}^{N} \int_\Omega e_{ij} \frac{\partial u}{\partial x_i} dx \quad (21)$$

Then, if u is a solution of the problem ($P_e;L;N$), it will check the following variational formulation:

$$\frac{\partial}{\partial t} < u, v > + a_0(t,u,v) = < P_c, c > \quad in \quad H^{-1}(\Omega)$$

$$for \quad all \quad v \in H_0^1(\Omega)$$

**Proposition 0.3.4** *$a0(t, ., .)$ satisfies the following tow assertions:*

1. *There exists a constant M* > 0 *such as:*

$$\left| a_0(t,u,v) \right| \le M \|u\| H_0^1(\Omega) \|v\| H_0^1(\Omega)$$

$$for \quad all \quad v \quad and \quad v \in H_0^1(\Omega)$$

Which means that $a_0$ is a continous biliniear form.

2. Coersivity in the following sens: there exists tow $\varepsilon_1$ and $\varepsilon_2 > 0$ such as:

$$a_0(t,u,u) + \varepsilon_1 \|u\|^2_{L^2(\Omega)} \ge \varepsilon_2 \|u\|^2_{H_0^1(\Omega)}$$

Proof

The measurability of $a0$ is a consequence of the continuity of ~ *eij ; eij* , u and v.

1 Continuity:

Let $u,v \in H_0^1(\Omega)$, from (21) we have:

Poincarre inequality implise that:

$$a_0(t,u,v) \le \sup_{i,j=1}^{N} \left| \tilde{e}_{ij} \right| N \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} +$$
$$\sup_{i,j=1}^{N} \left| e_{ij} \right| N^2 \|\nabla u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}$$

Poincarre inequality implise that:

$$\left| a_0(t,u,v) \right| \le A_0 \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}$$

For

$$A_0 = 2N \sup_{i,j=1}^{N} (\left| \tilde{e}_{ij} \right|, \lambda_1 N \left| e_{ij} \right|)$$

Which gives:

$$\left|a_0(t,u,v)\right| \le A_0[\|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}][\|\nabla v\|_{L^2(\Omega)} + \|v\|_{L^2(\Omega)}]$$

That we have:

$$\left|a_0(t,u,v)\right| \le A_0 \|u\|_{H_0^1(\Omega)} \|v\|_{H_0^1(\Omega)})$$

**Remark 0.3.5** *Constant A*0 *is maximum* .

2. Coersivity:

We decompose the form *a*0(*t; u; u*) in the following way:

$$a_0(t,u,u) = a_0^1(t) + a_0^2(t)$$

Where:

$$a_0^1(t) = \sum_{i,j=1}^{N} \int_\Omega \widetilde{e}_{ij} \frac{\partial u \partial v}{\partial x_i \partial xj} dx$$

$$and \quad a_0^2(t) = \sum_{i,j=1}^{N} \int_\Omega e_{ij} \frac{\partial u}{\partial x_i} u dx \quad (22)$$

-In one hand, integration by parts gives:

$$- a_0^2(t) \ge -\frac{N}{2} \sup_{i,j=1}^{N} \left\| \frac{\partial e_{ij}}{\partial xi} \right\|_{L^\infty} \|u\|^2{}_{L^2(\Omega)} \quad (23)$$

- In the other hand, we have to distinguish three situations:

If there is an isotropic condition, the tensor of thermic conductivity will be diagonal: $e_{ij} = 0$ for *i 6= j*.

So that

$$a_0(t,u,u) \ge -\frac{N}{2} \sup_{i,j=1}^{N} \left\| \frac{\partial e_{ij}}{\partial xi} \right\|_{L^\infty} \|u\|^2{}_{L^2(\Omega)} + \inf_{i,j=1}^{N} \left|\widetilde{e}_{ij}\right| \|u\|_{H_0^1(\Omega)}$$

We choose:

$$\varepsilon_1 = \frac{N}{2} \sup_{i,j=1}^{N} \left\| \frac{\partial e_{ij}}{\partial xi} \right\|_{L^\infty} \text{ and } \varepsilon_2 = \inf_{i,j=1}^{N} \left|\widetilde{e}_{ij}\right|$$

If the case of monotony condition according to all direction, which means: $(u_i, u_j \ge 0)$

$$a_0(t,u,u) \ge -\frac{N}{2} \sup_{i,j=1}^{N} \left\| \frac{\partial e_{ij}}{\partial xi} \right\|_{L^\infty} \|u\|^2{}_{L^2(\Omega)} + \inf_{i,j=1}^{N} \left|\widetilde{e}_{ij}\right| \|u\|^2{}_{H_0^1(\Omega)}$$

Then we take:

$$\varepsilon_1 = \frac{N}{2} \sup_{i,j=1}^{N} \left\| \frac{\partial e_{ij}}{\partial xi} \right\|_{L^\infty} \text{ and } \varepsilon_2 = \inf_{i,j=1}^{N} \left|\widetilde{e}_{ij}\right|$$

For the general case, we have:

$$a_0^1(t) \ge \inf_{i,j=1}^{N} \left|\widetilde{e}_{ij}\right| \|u\|^2_{H_0^1(\Omega)} + 2 \inf_{i,j=1}^{N} \left|\widetilde{e}_{ij}\right| \Sigma_{1<i<j<N} \int_\Omega \frac{\partial u \partial v}{\partial x_i \partial xj} dx \quad (24)$$

Since:

$$2\Sigma_{1<i<j<N} \int_\Omega \frac{\partial u \partial v}{\partial x_i \partial xj} dx \ge -\sum_{i=1}^{N} \int_\Omega \left(\frac{\partial u}{\partial x_i}\right)^2 dx \quad (25)$$

By (23) (24) and (25) we obtain:

$$a_0^1(t) \ge \inf_{i,j=1}^{N} \left|\widetilde{e}_{ij}\right| \|u\|^2_{H_0^1(\Omega)} + 2\Sigma_{1<i<j<N} \int_\Omega \frac{\partial u \partial v}{\partial x_i \partial xj} dx$$

$$a_0^1(t) \ge \inf_{i,j=1}^{N} \left|\widetilde{e}_{ij}\right| \|u\|^2_{H_0^1(\Omega)} - \inf_{i,j=1}^{N} \left|\widetilde{e}_{ij}\right| \sum_{i=1}^{N} \int_\Omega \left(\frac{\partial u}{\partial x_i}\right)^2 dx$$

Which gives that: $a_0^1(t) \ge 0$

Then:

$$a_0(t,u,u) + \frac{N}{2} \sup_{i,j=1}^{N} \left\| \frac{\partial e_{ij}}{\partial xi} \right\|_{L^\infty} \|u\|^2{}_{L^2(\Omega)} \ge 0$$

$$a_0(t,u,u) + \frac{N}{2} \sup_{i,j=1}^{N} \left\| \frac{\partial e_{ij}}{\partial xi} \right\|_{L^\infty} \|u\|^2{}_{L^2(\Omega)} + (1-\beta)\|u\|^2_{H_0^1(\Omega)} \ge \beta \|u\|^2_{H_0^1(\Omega)}$$

Where β is an arbitrary parameter, which verify:

$$0 < \beta < \frac{1}{2}$$

We obtain:

$$a_0(t,u,u) + \frac{N}{2} \sup_{i,j=1}^{N} \left\| \frac{\partial e_{ij}}{\partial xi} \right\|_{L^\infty} \|u\|^2{}_{L^2(\Omega)} \ge (1+2\beta) \|u\|^2_{H_0^1(\Omega)}$$

Then we take:

$$\varepsilon_1 = \frac{N}{2} \sup_{i,j=1}^{N} \left\| \frac{\partial e_{ij}}{\partial xi} \right\|_{L^\infty} \text{ and } \varepsilon_2 = 1 + 2\beta \text{ for } 0 < \beta < \frac{1}{2}$$

If we choose: $\varepsilon_1$ and $\varepsilon_2$ such as:

$$\varepsilon_1 = \frac{N}{2} \sup_{i,j=1}^{N} \left\| \frac{\partial \varepsilon_{ij}}{\partial x_i} \right\|_{L^\infty} \text{ and}$$

$$\varepsilon_2 = \sup(1+2\beta, \inf_{i,j=1}^{N} \left|\widetilde{e}_{ij}\right|) \quad (\beta \in \left]0, \frac{1}{2}\right[)$$

we have coersivity for all cases.

**Theorem 0.3.6** *The problem* ($P_{e,L,N}$) *admits a unique global solution.*

**Proof**

Using proposition **0:3:4,** we have continuety and coersivity of the bilinear form.

Thus theorem 1 and 2 of [4] make it possible to obtain the result.

**0.3.3 General case**

Now, we treat the problem (*Pe*) under the equivalent version ($P_{e,L,\lambda}$)

$$(P_{e,L,\lambda}) \begin{cases} \dfrac{\partial u}{\partial t} + L_t(u) = \lambda F_c(u) + P_c \quad in \quad \Omega x IR \\ u\big/_{\partial\Omega x IR^+} = 0 \\ u\big/_{\partial\Omega x\{0\}} = u_0 \end{cases}$$

**Theorem 0.3.7** *The problem* ($P_{e,L,\lambda}$) *admits a unique global solution.*

Proof of this theorem will need a whole of lemmas.

**Lemma 0.3.8** * *Fc checkes:*

*(a)* $F_c(x,t,u) \le F_1|u|$ almost every where in IR.

(b) $\dfrac{\partial F_c(x,t,u)}{\partial u} + F_2 \ge 0$ almost every where in $D_\infty$

(c) There exists $\delta_0 > 0$ such that:

$e^{-\delta t} F_c(x,t,e^{\delta t}u) \quad \in L^2(\Omega)$ for $\delta > \delta_0$.

**Proof**

According to the properties checked by F, it is enough to take:

$F_1 = \gamma_1 c_0^{-1}$ and $F_2 = \gamma_2 c_0^{-1}$

which prove the first and second points of the lemma.

Let u element of : $H_0^1(\Omega)$ then $e^{\delta t}u \quad \in H_0^1(\Omega)$

Thus

$$\left|F_c(x,t,e^{\delta t}u)\right| \le c\left|e^{\delta t}u\right|$$

That implises:

$$e^{-\delta t} F_c(x,t,e^{\delta t}u) \quad \in L^2(\Omega)$$

**Definition 0.3.9** An operator T is known as monontonous in J.Lions sens if :

$< Tv - Tw, v - w > \ge 0$ for all v,w in $H_0^1(\Omega)$

In the following, we take:

$\tilde{u} = e^{-\delta t}u$

We obtain:

$L_t(\tilde{u}) = e^{-\delta t} L_t(u)$

and then:

$$\frac{\partial \tilde{u}}{\partial t} + \partial\tilde{u} + L_t(\tilde{u}) = \lambda e^{-\delta t} F(x,t,e^{\delta t}\tilde{u}) + e^{\delta t} P_c$$

Let's put:

$F_1(x,t,\tilde{u}) = F_c(x,t,e^{\delta t}\tilde{u})$

Thus u solution of a problem $(P_{e,L,\lambda})$ if and only if u checks:

$$\frac{\partial \tilde{u}}{\partial t} + \partial\tilde{u} + L_t(\tilde{u}) = \lambda e^{-\delta t} F_c(x,t,e^{-\delta t}\tilde{u}) + e^{-\delta t} P_c$$

*Let* $b_0$ be *defined by:*

$$b_0(t,v,w) =$$
$$\delta \int_\Omega vw dx + a_0(t, v-w, v-w) - \lambda e^{-\delta t} \int_\Omega (F_1(x,t,v)w dx$$

We consider the variationel problem defined by:

$$\frac{\partial}{\partial t} < u, v > + b_0(t,u,v) = \left\langle e^{-\delta t} P_c, v\right\rangle$$

Then we define the operator $b_1$ by:

$$< b_1(t)u, v > = b_0(t,u,v)$$

**Lemma 0.3.10** The operator $b_1(t)$ depends mesurably on t, and is hemi-continous.

**Proof**

We must show that:

For any sequence $t_n \to 0$ we have $b_1(t_n) \to b_1(0)$

Indeed,

$< b_1(t_n)u, v > = b_0(t_n, u, v)$

$$= \delta \int_\Omega vw dx + a_0(t_n, v, w) - \lambda e^{-\delta t} \int_\Omega F_1(x, t_n, v) w dx$$

Proprieties of $a_0(t, u, v)$ and F permet to conclude.

**Proposition 0.3.11** It exists $\delta_0^*$ such that, for all $0 < \delta < \delta_0^*$ ; is monotonous.

**Proof**

While posing

$B = < b_1(t)v - b_1 w, v - w >$

Thus

$$B = \delta \int_\Omega (v-w)^2 dx - \lambda e^{-\delta t} \int_\Omega (F_1(x,t,v) - F_1(x,t,w)(v-w) dx$$
$$+ a_0(t, v-w, v-w)$$

What implies:

$$B \ge \delta \|u\|^2_{L^2(\Omega)} - \varepsilon_1 \|v-w\|^2_{L^2(\Omega)} - \varepsilon_2 \|v-w\|^2_{H_0^1(\Omega)} -$$
$$\lambda F_2 \|v-w\|^2_{L^2(\Omega)}$$

Let us choose for example:

$\delta_0^* = \varepsilon_1 + \lambda F_2$

**Proposition 0.3.12 I :** It exists $\tilde{\varepsilon} > 0$ such as :

$$\left\langle b_1(t)u, u\right\rangle \ge \varepsilon \|u\|^2$$

For all: $\varepsilon < \tilde{\varepsilon}$

**Proof**

We have:

$$< b_1(t)u, u > =$$
$$\delta \int_\Omega u^2 dx + a_0(u,u,t) - \lambda e^{-M} \int_\Omega F_1(x,t,u) u dx$$
$$\ge \delta \|u\|^2_{L^2(\Omega)} + \varepsilon_2 \|u\|^2_{H_0^1(\Omega)} - \varepsilon_1 \|u\|^2_{L^2(\Omega)} - \lambda F_2 \|u\|^2_{L^2(\Omega)}$$

The previous result gives:

$$< b_1(u), u > \ge (\delta - \varepsilon_1 - \lambda F_2) \|u\|^2_{L^2(\Omega)} + \varepsilon_2 \|u\|^2_{H_0^1(\Omega)}$$

Let

$\varepsilon^* = \delta - \varepsilon_1 - \lambda F_2$

We obtain:

$$< b_1(u), u > \ge \varepsilon^* \|u\|^2_{L^2(\Omega)} + \varepsilon_2 \|u\|^2_{H_0^1(\Omega)}$$

If we choose: $\tilde{\varepsilon} = \inf(\varepsilon_1^*, \varepsilon_2)$

We obtain the result.

**Proof of theorem**

**Lemma 0.3.9** propositions 0.3.10, 0.3.11 and theorem 1, 2, and 3 of [4] give the theorem **0:3:6.**

**Remark 0.3.13** For reasons of compatibility between **lemma 0.3.7** and proposition 0.3.10, we will choose: $\delta_0 < \delta_0^*$. Indeed, it is enough to choose $\delta_0 = \varepsilon_1$ and in this case:

$$\delta_0 = \varepsilon_1 < \delta < \delta_0^* = \varepsilon_1 + \lambda F_2$$

Where $\lambda F_2 \ne 0$ and in this case, we have $\delta \in \left]\varepsilon_1, \varepsilon_1 + \lambda F_2\right[$

**Conclusion:**

In this paper, we obtained the theorem 0.4.6 by two staps. In the first time, we have studied the linear case by constructing the operator $L_t$ and natural assumptions ( thermophysical nature of the data). For the nonlinear case, we exploited a variationnal formulation and application of the results of D.L for obtaining the existence and unicity.

In a future work, we hope to obtain the same result with dependence not only of space and time but also of the thermaleld.

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