Nonperturbative Analysis of the Harmonic Oscillator with a Time-Dependent Frequency

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We apply the Floquet theorem and the resonating averages method to solve the Schrödinger equation of the harmonic oscillator whose frequency varies periodically with time. We determine its Floquet states and its corresponding wave functions. The formalism provides an efficiency tool for analyzing the various transition probabilities as well as the uncertainty relation of the system. The effect of a periodic external force on the Floquet states and the two-photon algebra realization are shortly discussed.

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1. Introduction

In view of the importance that the coherent states treatment of the time-dependent harmonic oscillators (TDHO) has and the many applications that has in various areas of physics, notably in quantum optics, the description of radiations in a resonant cavity in presence of atoms while taking account of interactions with partitions has immediate relevance in the generation of the coherent states by external change of the harmonic oscillators mass and the frequency [1-6]. There exists various methods for solving the Schrödinger equation of such a system. The well-known among them is the time-dependent perturbation theory. A generalized invariant operator method has been developed and applied notably to the TDHO with damping, and in the presence of a perturbative force. It was also used for some special cases of general quadratic systems [7-10]. In particular, among them the generalized Floquet formalism [11-14] which has been developed and applied to nonperturbative approaches of physical systems to predict and give explanation of variety of phenomena, including chemical bond hardening analysing [15,16], multiple high order harmonic generation in intense laser field [17], selective excitation of molecular vibrational states using short laser pulses [18], etc. Moreover, the adiabatic invariance principle plays a major role in the determination of quantum states of quantum systems. Floquet theory is particularly a powerful tool in the case of a system submitted to time-periodic strong field such as in the case of laser fields and consequently for which the usual stationary states, solution of the Schrödinger equation when no external field acts disappear. The Floquet theorem proves that for quantum systems in the presence of a driving field periodic in time, there exists a canonical transformation which transforms the nonconservative time-dependent Schrödinger equation into a conservative one, which resolution needs well-known standard methods [19-21]: the adiabatic invariance principle is then invoked to give a physical interpretation to the derived quantum states. The system is assumed to reach new quantum states or Floquet states (or steady states) which constitute the most probable states at thermodynamic equilibrium. The previous transformation may be compared to the customary rotating-wave approximation [5, 22].

The aim of this work is to apply the Floquet formalism to study special cases of TDHO and the forced TDHO. Floquet theorem is used together with the so-called the resonating averages method (RAM).

The RAM provides a useful tool for constructing the
Floquet operators in a whole resonance zone at whatever approximation order. This paper is structured as follows.

We first present the principle outlines of these theoretical approaches, and we apply them to the TDHO which frequency varies according to a determined periodic law. Next, we express its Floquet states, and the corresponding wave functions. Then we compute the transition probabilities, we analyze the uncertainty relation and study the effect of a periodic external strength upon these states. Finally, the main two-photon algebra realization is carried out for illustrating the previous process and the results are briefly discussed.

2. Theoretical background

The Schrödinger equation of a quantum system with time-dependent Hamiltonian H(t) can be written in terms of the time-evolution operator U (t, t_o):

$$i\hbar \frac{dU(t, t_0)}{dt} = H(t)\, U(t, t_0) \qquad (1)$$

where

$$H(t) = H_o + \varepsilon H_1(t) \qquad (2)$$

and U(t, t_o) is such that

$$|\psi(t)\rangle = U(t, t_o)\, |\psi(t_o)\rangle \qquad (3)$$

H_o the Hamiltonian of the unperturbed system, and $H_1(t)$ the Hamiltonian of the interaction of amplitude ε. □□(t)> describes the system state at the time t.

In the case of a periodic interaction, the Floquet theorem asserts the existence of a couple of operators (R, T(t)), where R is a constant Hermitian operator, and T(t) is a periodic unitary operator of the same period as $H_1(t)$ [19, 20] so that the general form of the operator V(t), solution of Eq. (1), is given by

$$V(t) = T(t)\, e^{-iR(t-t_o)/\hbar} \qquad (4)$$

where

$$U(t, t_o) = V(t)\, V^{-1}(t_0) \quad \text{and} \quad U(t_o, t_o) = \mathbf{1} \qquad (5)$$

and

$$R = T^{-1} H T - i\hbar\, T^{-1} \frac{\partial T}{\partial t} \qquad (6)$$

The application of the unitary transformation T(t) on Eq. (1) gives rise to the so-called reduced representation of the system (i.e. reduction of Eq. (1)), whose resolution leads to canonical solutions of the time-dependent Schrödinger equation . Consequently, the Floquet states of the system are defined as being

$$|\psi_k(t)\rangle = e^{-i\lambda_k t/\hbar}\, T(t)\, |r_k\rangle \qquad (7)$$

where, {$|r_k\rangle$} are the eigenvectors of the operator R (in the basis of the proper states of H_o), corresponding to the eigenvalues λ_k.

These new quantum states constitute a complete set of bound states of the system and do not depend on the choice of the couple (R, T(t)). One shows otherwise that these are the most probable states of matter in the presence of a coherent radiation (i.e. the invariant adiabatic states). They generalize in fact the notion of customary stationary states of a conservative quantum system, when the nonconservative one is submitted to a sufficiently strong periodic field. Thus, , $|\psi(t)\rangle$ of the system is described by a linear combination of these Floquet states.

Let us insist in the fact that the Floquet theorem stipulates the existence of an infinite number of couples which are solutions of Eq. (1) and the reduced equation of the system. However the determination of R and T(t) is not always immediately settled, but it depends on the form of $H_1(t)$. In our case the choice of R and T(t) will be done by use of the resonating averages method (RAM) which avoids secular terms near the resonance zone. It permits to get, to a certain order in power of ε, an approximate solution of Eq. (1) [21, 23].

In the intermediate picture, the interaction Hamiltonian $H_I(t)$ is defined as

$H_I(t) = e^{iH_0 t/\hbar} H_1(t)(t)\, e^{-iH_0 t/\hbar}$

The principle of the RAM consists in separating $H_I(t)$ in two parts as follows [23]

$$H_I(t) = \overline{H}_I(t) + \frac{d\widetilde{H}_I(t)}{dt} \quad (8)$$

where, $\overline{H}_I(t)$ and $\widetilde{H}_I(t)$ are respectively the averaging part and the oscillating part of the interaction Hamiltonian.

The application of the above method to first order in ε gives,

$$i\hbar \frac{d\,{}^{(1)}V_I(t)}{dt} = \varepsilon \overline{H}_1(t)\,{}^{(1)}V_I(t) \quad (9)$$

and the fundamental performed solution ${}^{(1a)}U_I(t)$ is then

$${}^{(1a)}U_I(t) = \left[1 - \frac{i}{\hbar}\varepsilon \widetilde{H}_I(t)\right] {}^{(1)}V_I(t) \quad (10)$$

Hence, the solution ${}^{(1)}V_I(t)$ of Eq. (9) with a comparison of Eqs. (4) and (10) leads to determine the Floquet operators : ${}^{(1a)}T(t)$ and ${}^{(1a)}R$.

3. Application to the harmonic oscillator with a periodical frequency

In the specific case where the periodical frequency Ω (t) has the form,

$$\Omega^2(t) = \omega_0^2(1 + \varepsilon \cos 2\omega t) \quad (11)$$

The oscillator with a constant mass and time-periodic frequency such as Eq. (11), is an important special case of Mathieu oscillator which enables to interpret the quantum motion of the Paul trap ion (charged particle in a Paul trap) [24, 25].

The Hamiltonian H(t) of this system is given as follows [26-28]

$$H(t) = 2\hbar\omega_0 J_3 + [\varepsilon\hbar\omega_0 \cos 2\omega t](J_1 + J_3) \quad (12)$$

ω_0 is the frequency of the unperturbed oscillator and ε the perturbation amplitude taken as been very small (ε <<1).

J_1, J_2, J_3 are the SU(1,1) generators which are expressed in terms of the conjugates variables x and p on one hand and in terms of the creation and the annihilation operators according to

$$J_1 = \frac{\beta^2 x^2 - p^2}{4\hbar\beta} = \frac{1}{4}(a^2 + a^{+2}) \quad (13)$$

$$J_2 = -\frac{px + xp}{4\hbar} = \frac{i}{4}(a^2 - a^{+2}) \quad (14)$$

$$J_3 = \frac{\beta^2 x^2 + p^2}{4\hbar\beta} = \frac{1}{2}(a^+a + \frac{1}{2}) \quad (15)$$

where □ □ m□$_0$ and x and p satisfy the commutation relation $[x, p] = i\hbar$.

Inserting Eqs. (13) and (15) into Eq. (12) yields to write H_0 and $H_1(t)$ in Eq. (2) as

$$H_0 = \hbar\omega_0(a^+a + \frac{1}{2}) \text{ and } H_1(t) = \hbar\frac{\omega_0}{4}\cos(2\square t)\,[2a^+a + a^2 + a^{+2} + 1] \quad (16)$$

The application of the RAM, to this system, according to Eq. (8), leads to the following equations

$$\overline{H}_I(t) = \frac{\cos 2\omega o}{2} H_0 \text{ and } \widetilde{H}_I(t) = \eta(t)a^2 + \eta^*(t)a^{+2} \quad (17)$$

where

$$\eta(t) = \frac{\hbar\omega_0}{8}\frac{\omega \sin 2\omega t - i\omega_0 \cos 2\omega o}{\omega^2 - \omega_0^2} e^{-2i\omega_0 t} \quad (18)$$

and where $\eta^*(t)$ is the complex conjugate of η(t).

The expressions of the Floquet components to first order in ε are then

$${}^{(1a)}R = H_0 \text{ and } {}^{(1a)}T(t) = [1 - \frac{i\varepsilon}{\hbar} e^{-iH_0 t/\hbar}\, \widetilde{H}_I(t)\, e^{iH_0 t/\hbar}]\,{}^{(1)}V_I(t) \quad (19)$$

The solution of Eq. (9) is given in this case by

$${}^{(1)}V_I(t) = \exp[-iu(t)\{a^+a + \frac{1}{2}\}] \text{ where } u(t) = \frac{\varepsilon\omega_o}{4\omega}\sin 2\omega t \quad (20)$$

3.1. The Floquet states and wave functions

The eigenvalues of [1a] R and the corresponding eigenstates are respectively

$\lambda_k = (k + \frac{1}{2})\hbar\,\omega_0$ and $|\phi_k\rangle = e^{-i(k+\frac{1}{2})\omega_0 t}\,|k\rangle$; (k = 0, 1, . . ., n-1) (21)

The corresponding Floquet states are then written as

$$|\psi_k(t)\rangle = e^{-i(k+\frac{1}{2})(\omega_0 t+u)}\left[-d^*\sqrt{k(k-1)}\,|k-2\rangle + |k\rangle + d\sqrt{(k+1)(k+2)}\,|k+2\rangle\right] \quad (22)$$

with

$$d(t) = -\frac{i\varepsilon}{\hbar}\,\eta^*(t)\,e^{-2i\omega_0 t} \quad (23)$$

and k is an integer must be equal to 0, 1, 2, …, n-1 such as k + 2≤ n-1.
where n is a number of Floquet state of the system.
Let us note that these states form, to first order in ε, a complete orthonormal set since,
$|d|^2 \ll 1$. We agree the fact that these reduce to stationary states as $\varepsilon \to 0$, this can be interpreted as an adiabatic connect limit of the system.
The wave functions associated with these states are therefore,

$$\psi_k(x, t) = e^{-i(k+\frac{1}{2})(\omega_0 t+u)}\left[-d^*\sqrt{k(k-1)}\,\varphi_{k-2}(x) + \varphi_k(x) + d\sqrt{(k+1)(k+2)}\,\varphi_{k+2}(x)\right] \quad (24)$$

where $\varphi_k(x) = \left(\frac{\alpha}{\pi}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\,e^{-\alpha x^2/2}$

$H_k(\sqrt{l}\pm x)$ are the usual form of the simple oscillator wave functions, H_k are the Hermite polynomials of order k and $\alpha = m\frac{\omega_0}{\hbar}$.

3.2. Transition probabilities

The computation leads to distinguish between the two following cases

(i) For k and k' of different parity one obtains,

$$|\langle\psi_k|\,H_1|\psi_{k'}\rangle|^2 = 0 \quad (25)$$

There is absence of coupling effect between the Floquet states of the system.

(ii) For k and k' of the same parity one has,

$$|\langle\psi_k\,|\,H_1|\psi_{k'=k+2}\rangle|^2 = \frac{\hbar^2\omega_0^2\cos^2 2\omega\omega}{16}(k+1)(k+2)\,|1-(k+1)(k+2)d^{*2}|^2 \quad (26)$$

In this case the transitions are possible owing to the coupling effect. Theses transitions are due to contributions of the different terms of H_1 and are not interfering.

3.3. Uncertainty relation

With the help of some identities and by using the following relations,

$x = \sqrt{\frac{\hbar}{2m\omega_0}}\,(a + a^+)$; $p = -i\sqrt{\frac{m\omega_0\hbar}{2}}\,(a - a^+)$;

it is easily shown that the fluctuations in x and p are as well respectively

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega_0}}\,(2k+1)^{1/2}\left[1 + 4\mathrm{Re}(d) + 2(k^2+k+5)|d|^2\right]^{1/2} \quad (27)$$

$$\Delta p = \sqrt{\frac{m\omega_0\hbar}{2}}\,(2k+1)^{1/2}\left[1 - 4\mathrm{Re}(d) + 2(k^2+k+5)|d|^2\right]^{1/2} \quad (28)$$

It is clear that there is a squeezing of one operator at the expense of an increase in the fluctuation of the other operator. Thus the squeezing property of $|\psi_k(t)\rangle$ is apparent here.
Therefore, the dispersion of x and p product for all times shows that in any case the uncertainty principle is satisfied and has the minimum value only for the k = 0, given by

$$\Delta x\ \Delta p = \frac{\hbar}{2}\ [(1+10|d|^2)^2 - 16\ Re^2(d)]^{\frac{1}{2}} \quad (29)$$

It is easily seen that in the absence of the perturbation (ε = 0 i.e. d = 0) we recover the uncertainty relation of the simple harmonic oscillator and that the effect of the periodic change of □(t) involves a correction of the uncertainty relation with the parameter d(t).

3.4. The forced oscillator with periodic frequency

We would like to study in this subsection the effect on the previous system of an external periodical strength, given by [6]

$$f(t)=\begin{cases}0, t \le 0\\ \lambda \sin \omega t, t > 0, (0 \le \lambda \le 1)\end{cases} \quad (30$$

The corresponding potential is

$$V(t) = - f(t)\ x = - \lambda \sin(\omega t)\sqrt{\frac{\hbar}{2m\omega_0}}\ (a + a^+) \quad (31)$$

Under the action of this potential, the calculation of the transition probabilities between the Floquet states of the periodical frequency oscillator gives

(i) For k and k' of the same parity

$$|\langle\psi_k|\ V(t)\ |\psi_{k'}\rangle|^2 = 0 \quad (32)$$

transitions are therefore forbidden in this case.

(ii) For k and k' of different parities

$$|\langle\psi_k|\ V(t)\ |\psi_{k'=k+1}\rangle|^2 = [\sqrt{k+1}\ f(t)\sqrt{\frac{\hbar}{2m\omega_0}}\ ]^2\ |1+ 2d^*|^2;\ \ (k = 0, 1 \ldots, n-3) \quad (33)$$

and

$$|\langle\psi_k|\ V(t)\ |\psi_{k'=k+1}\rangle|^2 = [\sqrt{k+1}\ f(t)\sqrt{\frac{\hbar}{2m\omega_0}}\ ]^2\ |1- 2d^*|^2;\ \ (k = n -2) \quad (34)$$

These are clearly not vanishing in this case so that the steady states of the system become coupled in the presence of the periodical external strength. These results are in perfect agreement with those of Refs. [6, 10].

3.5. The two-photon algebra realization

For illustration, we will consider the smallest representation of the single mode two- photon algebra of the oscillator with periodic frequency [28], it consists of 4 × 4 matrices, which is particularly convenient for computing matrix elements of the Floquet operator's components and transition probabilities. This matrix representation is given explicitly in the proper basis of H_0 {|0>, |1>, |2>, |3>} as

$$^{(1a)}R = \hbar\omega_o \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 3/2 & 0 & 0 \\ 0 & 0 & 5/2 & 0 \\ 0 & 0 & 0 & 7/2 \end{pmatrix};$$

$$^{(1a)}T(t) = \begin{pmatrix} e^{-iu/2} & 0 & -d^*\sqrt{2}e^{-5iu/2} & 0 \\ 0 & e^{-3iu/2} & 0 & -d^*\sqrt{6}e^{-7iu/2} \\ d\sqrt{2}e^{-iu/2} & 0 & e^{-5iu/2} & 0 \\ 0 & d\sqrt{6}e^{-3iu/2} & 0 & e^{-7iu/2} \end{pmatrix} \quad (35)$$

where u(t) is given by Eq. (20) and d(t) by Eq. (23).

The eigenvalues of $^{(1a)}R$ and the corresponding eigenstates are again respectively

$$\lambda_k = (k + \frac{1}{2})\hbar\,\omega_0 \ \text{ and } \ |\phi_k\rangle = e^{-i(k+\frac{1}{2})\omega_0 t}\,|k\rangle;\ \ (k = 0, 1, 2, 3) \quad (36)$$

The wave functions, obtained from the Floquet states of this system are:

$$\psi_o(x, t) = (\frac{\alpha}{\pi})^{1/4} \exp[-i(u + \omega_0 t)/2]\ e^{-\alpha x^2/2}\ [1+ d(2\alpha x^2-1)] \quad (37\text{-a})$$

$$\psi_1(x, t) = (\frac{\alpha}{\pi})^{1/4} \exp[-i3(u + \omega_0 t)/2]\ e^{-\alpha x^2/2}\ x\sqrt{2\alpha}\ [1+ d(2\alpha x^2 -3)] \quad (37\text{-b})$$

$\psi_2(x, t) = (\frac{\alpha}{\pi})^{1/4} \exp[-i5(u + \omega_0 t)/2]\, e^{-\alpha x^2/2}$

$\sqrt{2}\ [-d^* + \alpha x^2 - \frac{1}{2}]$ (37-c)

$\psi_3(x, t) = (\frac{\alpha}{\pi})^{1/4} \exp[-i7(u + \omega_0 t)/2]\, e^{-\alpha x^2/2}$

$[\, 2x\sqrt{3\hbar} \pm (-d^* + \frac{\alpha}{3} x^2 - \frac{1}{2})]$ (37-d)

Hence, the allowed transition probabilities are given by

$$|\langle\psi_0 | H_1 | \psi_2\rangle|^2 = \frac{\hbar^2 \omega_0^2 \cos^2 2\omega\omega}{8} [1 + \frac{\varepsilon^2 \omega_0^2 (\omega^2 \sin^2 2\omega\omega - \omega_0^2 \cos^2 2\omega t)}{16(\omega^2 - \omega_0^2)^2}] \quad (38)$$

and

$$|\langle\psi_1 | H_1 | \psi_3\rangle|^2 = 3\frac{\hbar^2 \omega_0^2 \cos^2 2\omega\omega}{8} [1 + 3\frac{\varepsilon^2 \omega_0^2 (\omega^2 \sin^2 2\omega\omega - \omega_0^2 \cos^2 2\omega t)}{16(\omega^2 - \omega_0^2)^2}] \quad (39)$$

We have therefore an oscillation between the Floquet states of the same parity, at the frequency $\omega = \omega_0$. So that these states are coupled with the perturbation Hamiltonian. One can also compute the expectation values of x, p, x^2, p^2 by using the $\omega_0 x$, t equation and one finds the uncertainty product for all t as

$$\Delta x\ \Delta p \approx \frac{\hbar}{2} [1 + \frac{\varepsilon^2 \omega_0^2}{16(\omega^2 - \omega_0^2)^2} G(\omega, \omega_0, t)]^{\frac{1}{2}} \quad (40)$$

where

$$G(\omega, \omega_0, t) = 5\omega^2 \sin^2 2\omega t + \omega_0^2 \cos^2 2\omega t \quad (41)$$

The relations (40) and (41) show that the uncertainty principle $\Delta x\ \Delta p \geq \frac{\hbar}{2}$ is satisfied.

Let us argue that in the absence of the perturbation, we recover the uncertainty relation of the simple oscillator and that the effect of the periodic variation of □(t) involves an oscillating correction of the uncertainty relation. Besides the divergent character of this correction at a primary resonance frequency $\omega = \omega_0$ of the system is clear.

The coupled Floquet states under the action of the previous external strength are then,

$$|\langle\psi_0 | V(t) | \psi_1\rangle|^2 \approx \frac{\hbar}{2m\omega_0} [\lambda \sin(\omega t)]^2 [1 + \frac{1}{2} \frac{\varepsilon\omega_0^2 \cos 2\omega o}{\omega^2 - \omega_0^2}] \quad (42)$$

$$|\langle\psi_1 | V(t) | \psi_2\rangle|^2 \approx \frac{\hbar}{m\omega_0} [\lambda \sin(\omega t)]^2 [1 + \frac{1}{2} \frac{\varepsilon\omega_0^2 \cos 2\omega o}{\omega^2 - \omega_0^2}] \quad (43)$$

$$|\langle\psi_2 | V(t) | \psi_3\rangle|^2 \approx 3 \frac{\hbar}{2m\omega_0} [\lambda \sin(\omega t)]^2 [1 - \frac{1}{2} \frac{\varepsilon\omega_0^2 \cos 2\omega o}{\omega^2 - \omega_0^2}] \quad (44)$$

These quantities do not vanish in this case so that the steady-states of the system become coupled in the presence of the external action.

4. Conclusion

We have successfully applied the Floquet theorem and the RAM to the harmonic oscillator with periodical frequency (HOPF). Our study has shown the efficiency of these analysis tools. Our analysis indicates that the squeezing property is observed and the uncertainty relation is satisfied with regard to an oscillating correction term; this has immediate relevance to the periodical fact of frequency change of the oscillator. We have also as in the HOPF established comparable relations in the case of an oscillator with periodically varying mass ″humid oscillator″ [30]. We point out the similarity between the relations obtained in

both cases and as well as in the periodical damping oscillator. This formal equivalence is essentially due to the formal symmetry in the Hamiltonian of the kinetic and potential terms. Moreover, it is easy to prove, on the basis of our analysis, that an oscillator with both periodical mass and frequency and with damping presents the same quasi-coherence properties as each of the hereby studied oscillators. A supplementary excitation of the system with a periodic external strength has the effect of disqualifying transitions between states of the same parity and restores transitions between states of different parity (i.e. it shattered transitions between states of the same parity), this is due of the potential form associated to the time-dependent external force. Our results are in perfect agreement with those obtained by other authors using others methods for the same types of oscillators [6, 10, 29]. These results have been positively illustrated in the framework of the two-photon algebra representation. In any case, the selective excitation of the system defines the induced resonance change that has often been observed in the quasi-energy spectra.We are planning to extend the above approach to the second order approximation on one hand and to other quantum systems, on the other.

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