# Multifractal Properties of an Electrical Network in the Percolation Threshold

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We consider a square network formed of random electric resistances to the percolation threshold of bond $P_c = 0.5$. We calculate the distribution of current on the infinite cluster with a constant current and constant tension, using a method of gradient conjugated accelerated by LU decomposition. Then we study the multifractal spectrum of the current distribution. Our numerical results are in agreement with the results in literature. The form of the distribution function is nearly Gaussian with the existence of a long tail in the weak current zone. This study leads us to the following findings: the insufficiency of the multifractality to describe all scales of currents; the distribution of strong currents is well understood and it is multifractal, meanwhile the weak currents stays imperfectly known.We also obtain a part of the spectrum which describes the very weak currents that we assign to the currents scales.

**Key words:** Percolation, Multifractality, Fractality.

## 1. INTRODUCTION

The concept of the multifractality is a very strong formalism to the description of physics phenomena. Nevertheless, it can prove to be insufficient in certain case. One of those cases is the distribution of currents on the random resistance network (RRN). This paper is dedicated to the investigation on the problem of the current distribution.

We consider a network of resistance; one cuts bonds in a random way with a probability $p = p_c$. We associate to every bond a resistance $r = 1$ and we apply to the network a difference of potential $\Delta V = 1$. A distribution of currents settles, then, on the different bonds of the network. The complete information on the nature of this distribution is contained in the moments:

$$M(k) \equiv \langle L^k \rangle = \sum n(I) I^k \quad (1)$$

Where $n(I)$ the number of bonds characterized by an absolute value of the current $I$ and $\langle ... \rangle$ is the average on the different configurations.

The moment of order 0 of currents varies with the system size L according to:

$$M_0 = \left\langle \sum i^0 \right\rangle_I \propto L^{D_{bb}} \quad (2)$$

Where hooks $<...>_I$ designate an average on different realizations, with the same conditions on the limits: we feed up our system with a unit current. This constitutes the whole current, constant here.

The conductance of the network also follows a power law with $L$. Taking into account the boundaries conditions considered, we can write the resistance, global $R$, as the dissipated energy in the system:

$$R = \left\langle \sum i^2 \right\rangle_I = M_2 = L^{\left(t - \nu\left(d-2\right)\right)/\nu} \quad (3)$$

The noise function also gives a power law between the fourth moment of currents and the system size [1],

$$\begin{aligned} M_4 &= S_R R^2 I^4 \\ &\propto L^{-\left(b+2t-2\nu\left(d-2\right)\right)/\nu} \\ &\propto L^{-\left(b-2t+2\nu\left(d-2\right)\right)/\nu} \end{aligned} \quad (4)$$

Where $S_R = \frac{\langle \delta R \delta R \rangle}{R^2}$ is the fluctuation of the global resistance with $S_R \propto \Delta p^b$. The maximum current gives a contribution of the same type [2]. In the whole current, $x(n)$ goes to $1/\nu$, appreciable bonds fractal dimension, as n tends towards the infinity. We introduce an infinite set of exponents $x(n)$ such as:

$$M_n = 2\left\langle \sum |i|^n \right\rangle_I \propto L^{x(n)} \quad (5)$$

 

The different exponents of moments of the quantities studied have the following values [3; 4]:

$$
\begin{aligned}
x(0) &= D_B = 1.65 \pm 0.02 \\
x(1) &= \xi_R = -t/v + (d-2) = 0.978 \pm 0.01 \qquad (6) \\
x(2) &= b + 2\beta_L = -0.81 \pm 0.02 \\
x(3) &= -0.77 \pm 0.03 \\
x(4) &= -0.74 \pm 0.02
\end{aligned}
$$

Analysing these exponents, we note no linearly law for exponents of moments of the current distribution. This means that they are independent, and the knowledge of those exponents of two orders doesn't permit to deduce the others, of third order. Therefore all exponents are necessary to describe the distribution of currents completely in a random resistance network (RRN). Even in the case where the network is submitted to a difference of potential unit, the set of exponents $x(n)$ are not merely linear in $n$. We are in presence of multifractality, and one sometimes represents this set $x(n)$ by its Legendre transformation:

$$\alpha = \partial x(n) / \partial n$$
$$f(\alpha) = x(n) - n\alpha$$

## 2. MODEL AND NUMERIC SIMULATION

Our numeric treatment has been established using a random resistance network presented with the following features:

- It is a square network inclined to 45° containing $L$ sites on every side.
- A tension is applied to two sides of the network; the two others are linked through cyclic conditions.
- The network is degraded with a rate $1-p$. Says otherwise, we cut $(1-p)N$ bonds randomly (it is bonds that don't make pass current).

The slope of the network makes the number of bonds of network $N$ to be equal to $L' \times L'$ with $L' = 2L$. To avoid confusions with the right network; we consider a network of size $L'$.

In this network, resistances take two values: unity with a probability $p$ and infinite with a probability $1-p$.

## 3. THE RESEARCH OF THE INFINITE CLUSTER

The research of an infinite cluster on percolation structures is an essential step in the transport phenomena investigation. Such studies can lead to an insoluble algebraic equation system, with several unknowns. Then appears the necessity to develop an algorithm, allowing us to get the infinite cluster of a percolate network. In our work, we develop a dynamic method of analysing a cluster distribution on percolate structures. Indeed, this method uses a set of variables created with the progression of needs of the program. Otherwise, a technique to sweep of the network by row has been developed [5].

This method offers a duplicate favour: in one hand, it permits us to optimize the memory storage and then, big structures could be generated. In the other hand, it reduces calculation time by examining a reduced part through a "window" on the network.

## 4. APPLICATION OF KIRCHHOFF LAWS ON THE RRN

Now, we have to determine tensions on every site $i$ of the network in way to be able to determine the distribution of currents and multifractal spectrum then. To do so, it is necessary to solve the discrete Poisson equation of on the degraded network. We proceed as follows:

For every site $i$, "*the sum of currents entering to the site $i$ is equal to the sum of currents coming out of the site* ".

An example is given in the figure 1: on a site $i$ of the network we have:

$$i_1 + i_2 = i_3 + i_4$$

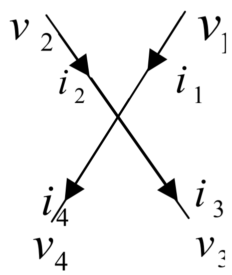


**Figure1**: the representation of a site $i$, with its four nearer neighbours sites.

We define $r_i^j$, the resistance between two sites $i$ and $j$ of a bond, and $V_i$, the tension on a site $i$, as:

$$\left(r_i^1\right)^{-1}(v_1 - v_i) + \left(r_i^2\right)^{-1}(v_2 - v_i) + \left(r_i^3\right)^{-1}(v_3 - v_i) + \left(r_i^4\right)^{-1}(v_4 - v_i) = 0$$

So the equation for every site $i$ is:

$$\sum_j \left(r_i^j\right)^{-1}\left(v_j - v_i\right) = 0$$

Where $j$ indicates the four sites that surround the site $i$.

Applying this equation to all sites of the resistance network, one get a system of linear equations:

$AV = B$

## 5. RESOLUTION OF THE EQUATION SYSTEM

Systems of linear algebraic equations, previously introduced, define some symmetrical and extremely sparse matrices. Iterative methods are, generally, preferred for this linear system types (obtaining of solutions with a controlled precision). In our case, we used the conjugated gradient method. Let's consider the linear system $AV = B$ of rank $n$. Taking an initial $x^0$, iterations of conjugated gradient are repeated until the maximum of the residual (noted $r^k$) get a value less than a given one. The speed of convergence of the algorithm, essentially, depends on the conditioning of the system matrix [6]. In order to get the best possible efficiency in our numerical study, we improved the method of resolution of the conjugated gradient while introducing the technique of the pre-conditionment. The method of the conjugated gradient with pre-conditionment is certainly the fastest general technique.

To do this, we use pre-conditionment to find matrices which are lower triangular $L$ and upper triangular $U$, as [6]: $A = LU - R = C - R$; $L$ and $U$ must be very sparse and $R$ must be as neighbour of *0* as possible. So the decomposition is said incomplete and $C = LU$ is a matrix of pre-conditionment.

## 6. THE DISTRIBUTION OF CURRENTS ON A RANDOM RESISTANCE NETWORK

On a random resistor network of size $L = 60$ the analysis of the current distribution takes place by the histogram method [7]. Two cases are considered:

Constant current

In this case, currents on every bond of the network have to be normalized by the sum of currents of the first row.

- Constant tension

The same treatment as before, but currents are not normalized by the sum of currents of the first row. The variation of the current distribution is represented in the figure 2.

**Results and discussion**

Results for constant current and constant tension are comparable and in good agreement with results of the literature [8]. Indeed, for the most frequent currents, the shape of the distribution function is nearly Gaussian with a dissymmetrical spreading toward the weak current zone. We see that this characteristic generates a scaling behaviour distinct of the strong current region.

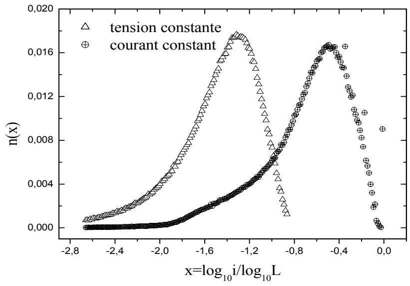


**Figure2** – the distribution of the currents for a square network of size L=60, in threshold of percolation.The histogram of the distribution of the variable $\log i$ are realised on the number of testesults obtained by parallel calculation on a cluster of PC' S for 15000 tests.

## 7. MULTIFRACTAL SPECTRUM OF THE CURRENT DISTRIBUTION

On the same network and with the same conditions as previously, the multifractal spectrum of the current distribution is represented, for the two treatments, on the figure 3.

The variation of multifractal spectrum to constant current with the variation of number of segments by a parallel treatment (MPI) is represented by the figure 4.

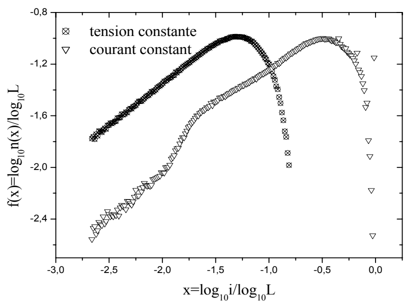


**Figure3**: the multifractal spectrum of the distribution of the currents $f(x)$ of a square network, in the threshold of percolation according to $x$, for a size L=60:spectrum obtained by parallel calculation on a cluster of PC' S for 15000 tests.

**Results and discussion**

In a general way, the figure 4 shows that the size of the segment is without important effect on the pace of the spectrum. However in the weak current region, fluctuations are important when the size of the segment increases. It is the number of points by segment that decreases when the number of segments grows. This effect is of a much more important than we are in the rare event region. We tried to know whether, in the literature, some similar behaviour exists. Effectively, the previous theoretical works [9] show that the pace of spectrums get explanation, in the interval of current values considered, by structures of blobs in scales.

On the figure 3, the region in the right describes the distribution of strong currents. It is concave and it is independent of the size of the system. It is well a multifractal behaviour.

On the other hand, the weak current region exhibits a multi - linear behaviour that indicates an abnormal behaviour in multifractal spectrum and that would be assigned to currents of scales [10].

## 8. CONCLUSION

In this work we led an investigation on the currents distribution on a square network in the percolation threshold. It allowed us to make clear some numeric elements. Results for constant current and constant tension are comparable and in good agreement with results of the literature. We succeeded to obtain a part of the spectrum describing the weak currents, assigned to currents of scales. So, these preliminary results are encouraging us to explore more very weak current regions and, then, to try to find out the dominant currents or the existing scaling laws.

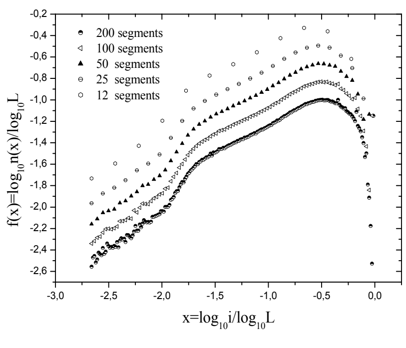


**Figure4***:* the representation of mulifractal spectrum f(x) according to x for different numbers of segments with a size L=60, for test number equal to 15000.

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