Potential of a wire bent into a circular shape Application to Saturn's Rings

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In this article, we develop the calculation of the potential generated by a homogeneous wire bent into a circular shape [1]. In a second time we develop a new method of perturbation. It gives an expression of the potential in terms of R, the radius of the circle. The potential is expressed as a sum of the Newtonian and a small term. The former will be considered as a perturbation. We give the orbits of a test particle in accordance with the initial conditions. Precession of perihelia or chaotic cases is proved [2]. In an accurate way, we must use a juxtaposition of such circular wires, in order to built a two dimensional disc
Key words: Potential-Legender Polynoms-Rings-Saturn.

I. INTRODUCTION.

The irregular shapes of many celestial bodies [3] have gained a great interest during the last decades. Their physical and geometrical studies require an accurate knowledge of the potential generated by them [4].

In our study, we develop the method of calculation of the potential generated by a circular wire in a point located at the plane of the wire. The result is given directly by a series expansion in terms of R the radius of the wire and his total mass. We are interested to the points outside the circle. To complete the analytical expression we use a new method of perturbation. It consists to establish the deviation from the Newtonian potential were the orbit is an ellipse in case of negative total energy.

II. POTENTIAL GENERATED BY A CIRCULAR WIRE.

We consider a circular homogeneous wire of radius R with total mass M and constant linear specific mass λ. The potential generated at the point M by an infinitesimal mass dm located at P is given by:

$$dV = -G\frac{dm}{PM} \qquad (1)$$

G stands for the universal gravitational constant.
We take Oxy as the plane of the circle and O his centre of mass.

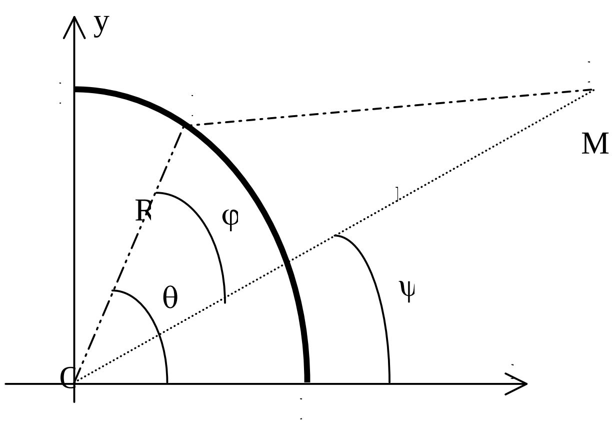

Figure 1
Circular wire

$$\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP} = (r\cos\psi - R\cos\theta)\,\vec{i} + (r\sin\psi - R\sin\theta)\,\vec{j}$$

The distance between P and M is then:

$$PM = \sqrt{R^2 + r^2 - 2rR\cos(\theta - \psi)}$$

with dm = $\lambda R d\theta$, (1) is written

$$V = -G\lambda R\int_0^{2\pi}\frac{d\theta}{\sqrt{R^2 + r^2 - 2rR\cos(\theta - \psi)}}$$

φ, the angle between OP and OM is given by $\varphi = \theta - \psi$, then

$$V = -\frac{G\lambda R}{\sqrt{R^2+r^2}}\int_{-\psi}^{2\pi-\psi}\frac{d\varphi}{\sqrt{1-\frac{2rR}{R^2+r^2}\cos\varphi}}$$

The point M is outside of the circle and located far away, r is then greater than R. We can expand in series by putting $\varepsilon = -2rR/(R^2+r^2)$.

To the second order in ε, we have:

$$V = -\frac{G\lambda R}{\sqrt{R^2+r^2}}\int_{-\psi}^{2\pi-\psi}\left(1-\frac{\varepsilon}{2}\cos\varphi + \frac{3\varepsilon^2}{8}\cos^2\varphi\right)d\varphi$$

The result of this integral is independent of ψ. We can take M along the x-axis. It yields then:

$$V = -\frac{2\pi G\lambda R}{\sqrt{R^2+r^2}}\left(1+\frac{3}{16}\varepsilon^2\right)$$

$$V = -\frac{GM}{\sqrt{R^2+r^2}}\left(1+\frac{3r^2R^2}{4(R^2+r^2)^2}\right)$$

Finally by expanding in terms of R/r:

$$V = -\frac{GM}{r} - \frac{GMR^2}{4r^3}$$

The potential is expressed in terms of R, the radius of the ring

V is viewed as two parts, one consist of the keplerian case, while the other summarise the perturbation. We write:

$$V(r) = V_0(r) + V_1(r) \qquad (2)$$

With:

$$V_0(r) = \frac{A}{r} \quad \text{and} \quad V_1(r) = \frac{B}{r^3}$$

Where $A = -GM$; $B = -GMR^2 = AR^2$

$$V(r) = \frac{A}{r} + \frac{B}{r^3}$$

III. PERTURBATION METHOD OF RESOLUTION.

The first term of the total potential is considered as the main part. It corresponds to the well-known keplerian case. The second one will be examined as a term of a small perturbation. We can, consequently take $\delta V = B/r^3$.

In the case of the presence of $V_0(r)$ only, the solution is given by Danby [5]:

$$\frac{p}{r} = 1 + e\cos\varphi \qquad (3)$$

In which

$$p = \frac{h^2}{mA} \quad \text{and} \quad e = \sqrt{1+\frac{2Eh^2}{mA^2}}$$

Where p: Semi-latus.
e: Eccentricity .
h: Angular momentum.
E: Total energy.
M: Test particle mass.

The periodic orbits interest us, so E is negative (E < 0) which corresponds to an eccentricity e lower than the unit (e < 1).The orbit is then an ellipse (Figure 1).

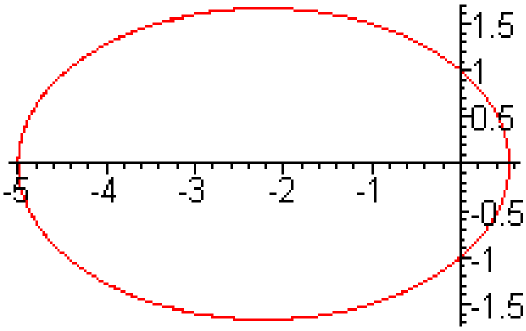

Figure 2
Keplerian orbit

The differential equation of motion is given by:

$$\frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + V(r) = E$$

with $mr^2\dot{\varphi} = h$ the constant of motion.

We deduce from the last two equations:

$$\varphi = \int \frac{h.dr}{r^2\sqrt{2m\left(E-\frac{h^2}{2mr^2}-\frac{A}{r}-\frac{B}{r^3}\right)}} \qquad (4)$$

We have obviously:

$$\varphi(r) = \varphi_0(r) + \delta\varphi(r) \qquad (5)$$

where $\varphi_0(r)$ corresponds to the keplerian case.

$$\varphi_0(r) = \int \frac{h.dr}{r^2\sqrt{2m\left(E-\frac{h^2}{2mr^2}-\frac{A}{r}\right)}} \qquad (6)$$

Expression of $\delta\varphi(r)$:

$$\delta\varphi(r) = \int \frac{Bh.dr}{r^5\sqrt{2m\left(E-\frac{h^2}{2mr^2}-\frac{A}{r}\right)}} \quad (7)$$

(7) is obtained by expanding (4) in Taylor series. If B is very small, we have

$$\varphi(r) = \varphi_0(r)$$

and then (3).

$$\delta\varphi(r) = \frac{\partial}{\partial h}\int \frac{\sqrt{2m}.B.dr}{r^3\sqrt{2m\left(E-\frac{h^2}{2mr^2}-\frac{A}{r}\right)}} \quad (8)$$

If we shift to the φ variable instead of r, we get:

$$\delta\varphi(r) = -m^2AB\frac{\partial}{\partial h}\left(\frac{1}{h^3}\right)\int(1+e\cos\varphi)d\varphi \quad (9)$$

The substitution of (5) into (3) gives:

$$\frac{p}{r} = 1+e\cos\varphi + e\sin\varphi.\delta\varphi \quad (10)$$

After a laborious calculation we arrive by (9) and (10) to the final expression (see appendices)

$$\frac{p'}{r} = 1+e'.\cos\lambda\varphi + f.\cos 2\varphi \quad (11)$$

With: $p' = p\left(1+\zeta\frac{e^3-2e^2+5e-1}{2e}\right).$

$$e' = e\left(1+\zeta\frac{e^4-2e^3-e^2-2}{2e^2}\right).$$

$$\lambda = 1+3\zeta\,.$$

$$\zeta = -\frac{m^2AB}{h^4}\,.$$

$$f = -\zeta\frac{e^3+2e^2+e+1}{2e}\,.$$

IV. INTERPRETATION:

From equation (3) we have the figure 2, which correspond to an elliptic orbit. This is a regular periodic orbit. The potential corresponding to is A/r.
To the second order we established the equation (11). From different initial conditions, we reach many different cases as in figures 3, 4 , 5 and 6.
Figure 3 correspond to a precession of the perihelia

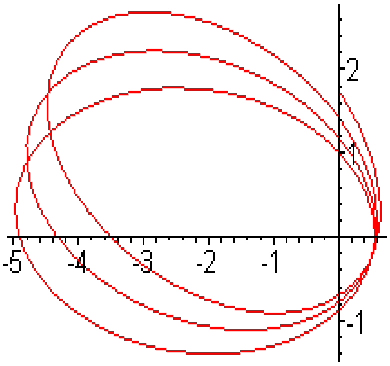

Figure 3Orbit with precession of the perihelia

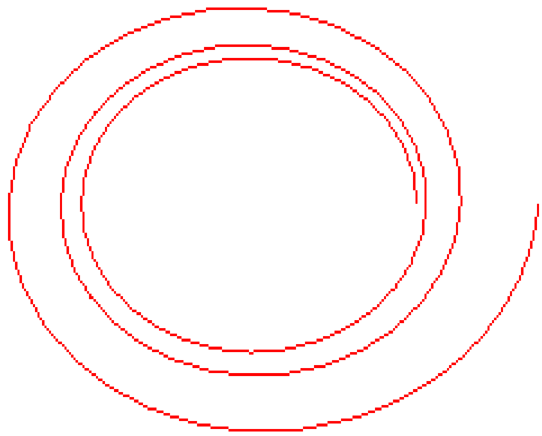

Figure 4 corresponds to a rotation from the initial ellipse to a location far away from the centre.

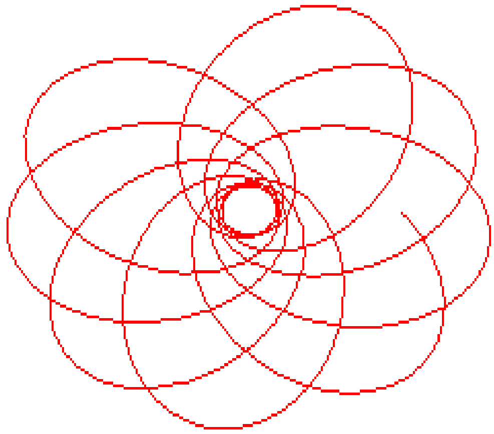

Figure 5

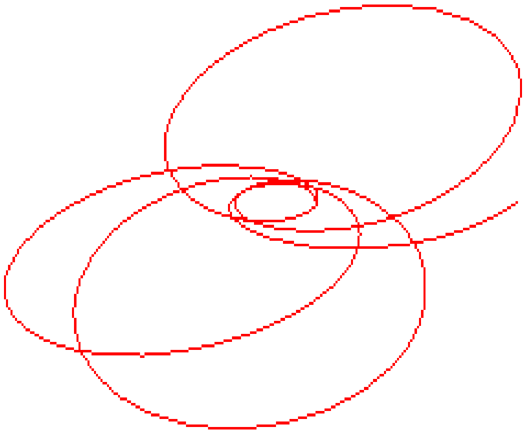

Figure 6

Figure 5 and 6 correspond to a disorder, which could reach a chaotic case.

V. CONCLUSION AND PERSPECTIVES:

In this research we gave a new method of perturbation to simplify the potential generated by a wire bent into a circle. We established the expression as a perturbation to the keplerian case. With the orbits we gave some conjectures to different cases of evolution of a test particle orbiting far away from the circle.
This situation is a realistic case of the interaction of particles with Saturn rings. We will give a deep study in a near future.
In the future, we plan to study a two dimensional case. This will correspond to a thin disc with constant density, as for the rings of Saturn.

[1]Gianni Pascoli 2000. Astronomie Fondamentale. Dunod.

[2] Murray and Dermott . 1999. Solar System Dynamics. Cambridge.

[3] Przemyslaw B. and Slawomir B. Celest. Mech. And Dyn. Astron. **86:** 131-141, 2003.

[4] Kellog. 1954. Foundation of Potential Theory. MacRobert Spherical Harmonics.

[5] Danby J.M.A. 1992.Fundamentals of Celestial Mechanics.Willmann-Bell.