Contribution of magnetic field lines stochasticity into anomalous transport in a tokamak

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The stochastic magnetic field lines behavior has been studied. We have developed a numerical technique in order to study their transition from partial to global stochasticity. Also, we have introduced a diffusion coefficient model (D_s) for to simulate the magnetic field lines diffusion though magnetic surfaces. The non-Gaussian dynamics of the field lines has been analyzed by calculating the kurtosis parameter.

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I. INTRODUCTION

Studies in the area of the controlled fusion in tokamak have shown that the matter and energy confinement depends strongly on the magnetic field structure. Magnetic surface destruction and formation of stochastic region of field lines resulting from magnetohydrodynamic instabilities lead to plasma confinement degradation [1]. Chaotically magnetic field lines diffusion through magnetic surfaces contribute to anomalous transport in tokamak by field structure perturbation. This perturbation leads to destruction of magnetic surface which assure confinement and to creation of magnetic islands surrounded by chaotic zones. The particles are then transported along chaotic magnetic field lines linking the central plasma region to the plasma edge [1-3].

In general magnetic field lines equations can be written in the Hamiltonian form [2,4,5] and their stochastic behavior can be studied correctly using the Poincaré section that can be described by discrete mapping [4-6].

The diffusion study in a confined plasma has been done by several authors [5,7-9]. Our contribution consists to study magnetic field lines diffusion through magnetic surfaces using a model that we have introduced and to analysis their non-Gaussian dynamics.

In the section II, we introduce for to study the stochastic behavior of the magnetic field lines the discrete mapping. In the section III, we show how one can detect, with a numerical method that we have elaborate, the transition from partial to the global stochasticity. In the section IV, we propose a diffusion coefficient model (D_s) that improves that of Mendonça [5] for values of the parameter of stochasticity K great than K_c ($K_c = 0.95$ is the Chirikov constant) and for which the ratio D_s/D_{QL} tends to the unity when K becomes very large accordingly to theoretical estimates [10]. In the section V, we study a non-Gaussian dynamics of the magnetic field lines by calculating, analytically using the Fourier technique and numerically the kurtosis parameter.

II. THE MAGNETIC FIELD LINES DESCRIPTION

In a tokamak, the magnetic coordinates at any position r are the toroidal flux function $\psi(r)$, the toroidal angle $\zeta(r)$ and the poloidal angle $\theta(r)$. The magnetic field can be written in the contravariant form as

$$\vec{B} = \vec{\nabla}\psi \wedge \vec{\nabla}\theta + \vec{\nabla}\zeta \wedge \vec{\nabla}\psi_p \tag{1}$$

where $\psi_p = \psi_p(r)$ is the poloidal flux. The field lines equations are

$$\frac{\partial\theta}{\partial\zeta} = \frac{\partial\psi_p}{\partial\psi}$$
$$\frac{\partial\psi}{\partial\zeta} = -\frac{\partial\psi_p}{\partial\theta} \tag{2}$$

To describe the stochastic behavior of the magnetic field lines, the Poincaré section can be used. For a perturbed magnetic field, the Poincaré representation of each magnetic field line in the plane $\zeta = 0 \pmod{2\pi}$ is described by the discrete mapping

$$I_{k+1} = I_k + \sum_m K_m \sin(m\theta_k)$$
$$\theta_{k+1} = \theta_k + I_{k+1} \tag{3}$$

where $I_k = 2\pi\psi_k$ and K_m are the parameters of stochasticity. The simple form of this mapping is the standard mapping [4,11] which correspond to $K_m = K\,\delta_{1m}$ (δ is the Kroenecker symbol).

III. TRANSITION TO THE GLOBAL STOCHASTICITY

A. Method of stochastic magnetic field lines determination

We know that if the perturbation applied on the magnetic field lines increases [9], the stochasticity destroy surfaces and region near separatrix called thin regions are first to become stochastic. If the fluctuation amplitude continues to increase, regular magnetic surfaces are gradually destroyed then the global stochasticity is reached. The transition from partial to global stochasticity takes place when the last KAM curve destroys [4,12]. We propose a method for to detect the curves destruction threshold. We utilize a computer progression that consist in the steady point by point of a magnetic field line. If for a given point $M(\psi_M, \theta_M)$ on the line L, we localizes an other point $N(\psi_N, \theta_N)$ on the same line. With field lines perturbation, the point N leaves the path and rest in a vicinity $V(M, \varepsilon)$ centered in M and of radius ε. If the MN module is different to zero when ψ_M tends to ψ_N

$$\lim_{\psi_M \to \psi_N} |MN| \neq 0 \tag{4}$$

we tell that the line is stochastic. If all the field lines verify the condition (4), then we have a global stochasticity.

B. Transition

When majority of magnetic field lines are stochastic a phenomenon of diffusion will take place. For the standard mapping, this condition is satisfied for $K \rangle K_c$. In the purpose to find the transition point, we consider any two points (ψ_1, θ_1) and (ψ_2, θ_2) belonging to the same magnetic surface supposed perturbed. If for $\psi_1 \neq \psi_2$ and $\theta_1 = \theta_2$, we speak of a closed contour beginning (magnetic island). In our investigation, we have taken 1000 field lines distributed on the interval [1, 10], the transition to global stochasticity is made when 100 per cent of the field lines satisfy the condition above. By applying this method for the standard mapping, we have find the Chirikov constant K_c from the figure 1.

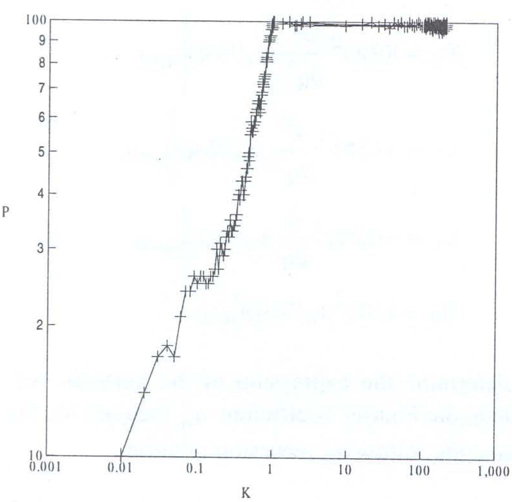

Figure 1: Transition from partial to global stochasticity. The per cent of stochastic field lines P calculated numerically for the mapping standard as function of the parameter K.

We have also calculated in the case of the generalized mapping to the order 2 given by [13]

$$\begin{aligned} I_{k+1} &= I_k + K_1 \sin(\theta_k) + K_2 \sin(2\theta_k) \\ \theta_{k+1} &= \theta_k + I_{k+1} \end{aligned} \tag{5}$$

the couples (K_1, K_2) from which we have the transition. (K_1, K_2) are the parameter of stochasticity of the generalized mapping. The figure 2 corresponds to a set of couples corresponding to the transition of magnetic surfaces to the stochastic state.

IV. MODEL OF DIFFUSION COEFFICIENT

The theoretical diffusion coefficient is

$$D = \lim_{t \to \infty} \frac{\left\langle (\Delta\psi)^2 \right\rangle}{2} \tag{6}$$

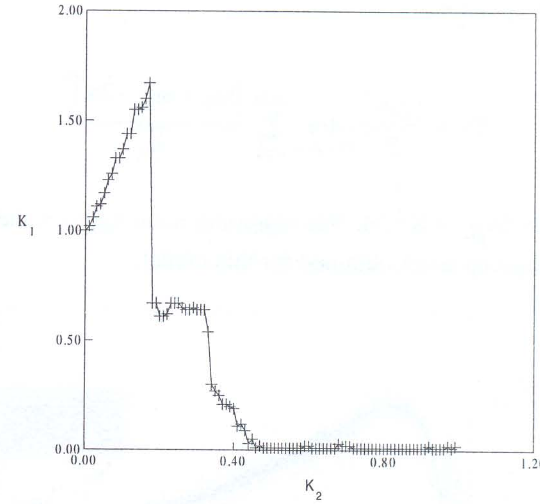

Figure 2: Transition partial-global stochasticity for the generalized mapping. The per cent of stochastic field lines P calculated numerically versus the parameter K for 1000 field lines.

where $\Delta\psi$ is the radial deviation of field line induced by the perturbation. Since this coefficient is insensitive to the transition from partial to global stochasticity, Mendonça suggested a diffusion coefficient

$$D_M = \frac{(\Delta I)^2}{2} \lim_{N \to \infty} \sum_{i=1}^{i=N} \frac{1}{L_i} \tag{7}$$

where L_i is the stochastic length and $\Delta I = 2\pi$. L_i is a number of iteration necessary to leave the interval $[0, 2\pi]$, i.e. $|\psi_f^i - \psi_0^i| \geq 2\pi$. When the line leave the interval $[0, 2\pi]$, we tell that it's a stochastic line and it's a trapped one in the opposite case. For trapped lines L_i are infinite then do not contribute to diffusion. Therefore, there exists a value $K = K_c$ from which lines begin to diffuse. The Mendonça model is valid only for values of the parameter of stochasticity K close to K_c. Therefore it is necessary to introduce a model that improves the Mendonça model. When $K \rangle\rangle K_c$, all lines leave the interval $[0,2\pi]$ after one or two iterations i.e. $L_i = 1$ or 2. In the Mendonça sense, all this lines are the same. Thus they have the same contribution in the diffusion coefficient $D_M\ (L_i = 1)$ what explains the saturation that appear with this model [5]. In fact, the magnetic field lines present an other aspect very important whose we had to take account : two lines that leave the interval $[0, 2\pi]$ at $L_i = 1$ make it with two different jumps $\Delta\psi_f^1 \neq \Delta\psi_f^2$. We have shown by our simulations that each diffusion line is specified by its stochastic length L_i and depends also on the factor $|\psi_f^i - \psi_0^i - 2\pi|^s$ where s is equal to 2,3. The empirical diffusion coefficient D_s verifies the condition

$$\lim_{K \to \infty} \frac{D_s}{D_{QL}} = 1 \tag{8}$$

D_s is given by

$$D_s = \frac{(\Delta I)^2}{2} \lim_{N\to\infty} \sum_{i=1}^{i=N} \frac{\left|\psi_f^i - \psi_0^i - 2\pi\right|^s}{L_i} \qquad (9)$$

where $D_{QL} = K^2/4$. We represent in the figure 3, the simulations result obtained for this model.

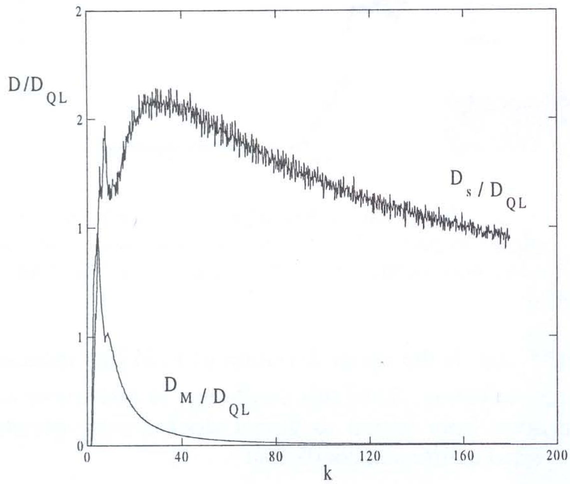

Figure 3: The ratio D_s/D_{QL} calculated numerically with $\Delta I=2\pi$, N =1000 field lines and s=2.3 as a function of the parameter K. The ratio D_M/D_{QL} calculated numerically with Mendonça model.

V. NON-GAUSSIAN DYNAMICS

In this section, we study the non-Gaussian behavior of stochastic magnetic field lines that are described by the standard mapping. We have then calculate the parameter kurtosis

$$\text{kurt} = \frac{\left\langle (\Delta I)^4 \right\rangle}{\left\langle (\Delta I)^2 \right\rangle^2} \qquad (10)$$

where $\Delta I = I - I_0$. For a non-Gaussian dynamics this parameter is different to 3 (i.e. kurt $\neq 3$). In the opposite case, i.e. kurt $= 3$. we have a Gaussian behavior.

$\left\langle (\Delta I)^4 \right\rangle$ and $\left\langle (\Delta I)^2 \right\rangle$ can be expressed in term of the conditional probability density w

$$\left\langle (\Delta I)^i \right\rangle = \int w(I, \theta, n / I_0, \theta_0, 0) \times (I - I_0)^i \, dI \, dq \qquad (11)$$

w gives the density of probability to found the system after n iteration at the state (I, θ) with the initial state is (I_0, θ_0) and satisfy the following recursion property

$$w = \int w(I, \theta, n / I', \theta', n-1) \times w(I', \theta', n-1 / I_0, \theta_0, 0) \, dI' d\theta' \qquad (12)$$

Expanding w in a Fourier series in θ and in a Fourier integral in I

$$w = \sum_m \int dq \; a_n(m, q) \exp(im\theta + iqI) \qquad (13)$$

$\left\langle (\Delta I)^4 \right\rangle$ and $\left\langle (\Delta I)^2 \right\rangle$ can be written as

$$\begin{aligned} \left\langle (\Delta I)^4 \right\rangle &= E_4 - 4I_0 E_3 \\ &\quad + 6{I_0}^2 E_2 - 4{I_0}^3 E_1 + {I_0}^4 E_0 \\ \left\langle (\Delta I)^2 \right\rangle &= E_2 - 2I_0 E_1 + {I_0}^2 E_0 \end{aligned} \qquad (14)$$

with $E_i = \int w I^i \, d\theta \, dI$. Introducing the eq. (13) in the expression of E_i, integrating over θ and using properties of the Fourier transform, we obtain

$$\begin{aligned} E_4 &= (2\pi)^2 \frac{\partial^4}{\partial q^4} a_n(0, q)\Big|_{q=0} \\ E_3 &= i(2\pi)^2 \frac{\partial^3}{\partial q^3} a_n(0, q)\Big|_{q=0} \\ E_2 &= -(2\pi)^2 \frac{\partial^2}{\partial q^2} a_n(0, q)\Big|_{q=0} \\ E_1 &= -i(2\pi)^2 \frac{\partial}{\partial q} a_n(0, q)\Big|_{q=0} \\ E_0 &= (2\pi)^2 a_n(0, q)\big|_{q=0} \end{aligned} \qquad (15)$$

To determine the expression of the kurtosis, we need to calculate the Fourier coefficient a_n (see ref. 4). For $n > 0$, we have the following recursion relation

$$a_n(m_n, q_n) = \sum_{l_n} J_{l_n}(K|q_{n-1}|) \times a_{n-1}(m_{n-1}, q_{n-1}) \qquad (16)$$

with

$$\begin{aligned} m_n &= m_{n-1} - l_n \operatorname{sgn}(q_{n-1}) \\ q_n &= q_{n-1} - m_n \end{aligned} \qquad (17)$$

The J_l is the Bessel function. Iterating n times the eq. (16) yields a_n in term of a_0

$$a_n(m_n, q_n) = \sum_{l_n, \ldots, l_1} J_{l_n}(K|q_{n-1}|) \times \ldots J_{l_1}(K|q_0|) \, a_0(m_0, q_0) \qquad (18)$$

where $a_0(m, q) = \frac{1}{(2\pi)^2} \exp(-im\theta_0 - iqI_0)$.

The set $\{l_n, \ldots, l_1\}$ define through (17) a path in the Fourier space (m, q). According to (15), the paths have to

take end at the point $m_n = 0$ and $q_n = 0$. The arguments of the Bessel functions are $K|q_i|$. Thus for K large, the dominant terms have $q_i \to 0$. In this case, the dominant term will be that having all $l_i = 0$, what corresponds to a path that remains during n steps at the origin. For this path,

$$a_n = \frac{1}{(2\pi)^2}[J_0(Kq)]^n \exp(-iqI_0) \qquad (19)$$

We have then

$$\left\langle(\Delta I)^4\right\rangle = \frac{3}{4}(n^2 - \frac{n}{2})K^4 + 12\,nK^2I_0^2 + 16I_0^4$$
$$\left\langle(\Delta I)^2\right\rangle = \frac{1}{2}nK^2 + 4I_0^2 \qquad (20)$$

To calculate the corrections to $\left\langle(\Delta I)^4\right\rangle$ and $\left\langle(\Delta I)^2\right\rangle$, we are going to suppose that the paths leave the origin. There is four three step paths what correspond to ($m_1 = 1$, $m_2 = -1$), ($m_1 = 2$, $m_2 = -2$) and their rotation by 180° which are paths that have respectively ($m_1 = -1$, $m_2 = 1$) and ($m_1 = -2$, $m_2 = 2$). We obtain

$$\left\langle(\Delta I)^4\right\rangle = \frac{3}{4}n^2K^4(\frac{1}{4} - J_2(K))$$
$$3nK^4(-\frac{1}{8} + 4J_2(K) + J_4(2K))$$
$$+12nK^2I_0^2(1 - 2J_2(K)) + 16I_0^4$$
$$\left\langle(\Delta I)^2\right\rangle = nK^2(\frac{1}{2} - J_2(K))$$
$$+2K^2J_2(K) + 4I_0^2 \qquad (21)$$

where we have combined the results given by a three step paths with these obtained for a path that remain at the origin n steps.

We turn now to the four step paths that contribute in the calculation of $\left\langle(\Delta I)^4\right\rangle$. We have computed 8 paths

- $m_1 = 1$, $m_3 = \pm 1$
- $m_1 = 1$, $m_3 = \pm 3$
- $m_1 = 2$, $m_3 = \pm 2$
- $m_1 = 3$, $m_3 = \pm 1$

We have also considered the five step paths that have the following property $m_1 = -m_2$ and $m_3 = -m_4$. This paths correspond to

- $m_1 = 1$, $m_4 = 1$
- $m_1 = 2$, $m_4 = 2$
- $m_1 = 1$, $m_4 = -1$
- $m_1 = 1$, $m_4 = 2$
- $m_1 = 2$, $m_4 = 1$

we have not mentioned here paths obtained by the 180° rotation of this paths (four and five steps). We have only multiplied by 2 the contribution of paths that we have given since a path and the path obtained by rotation 180 give the same contribution to a_n. The expression of $\left\langle(\Delta I)^4\right\rangle$ resulting from the contribution of all paths and for the case of n large is

$$\left\langle(\Delta I)^4\right\rangle = 3n^2K^4[\frac{1}{4} - J_2(K)$$
$$-J_1^2(K) + J_2^2(K) + J_3^2(K)]$$
$$+3nK^4[-\frac{1}{8} + 4J_2(K)$$
$$+6J_4^2(K) - 6J_2^2(K) - 6J_3^2(K)$$
$$+J_4(2K) + J_2^2(2K) + J_6^2(2K)$$
$$+J_4^2(2K) - 2J_1(K)J_5(3K)$$
$$-2J_5(K)J_7(3K) - 2J_2(K)J_4(2K)] \qquad (22)$$

As the same $\left\langle(\Delta I)^2\right\rangle$ has in this case the following expression

$$\left\langle(\Delta I)^2\right\rangle = nK^2(\frac{1}{2} - J_2(K)) \qquad (23)$$

Finally the kurtosis is

$$\mathrm{kurt} = \Big\{3n^2K^4[\frac{1}{4} - J_2(K)$$
$$-J_1^2(K) + J_2^2(K) + J_3^2(K)]$$
$$+3nK^4[-\frac{1}{8} + 4J_2(K)$$
$$+6J_4^2(K) - 6J_2^2(K) - 6J_3^2(K)$$
$$+J_4(2K) + J_2^2(2K) + J_6^2(2K)$$
$$+J_4^2(2K) - 2J_1(K)J_5(3K)$$
$$-2J_5(K)J_7(3K) - 2J_2(K)J_4(2K)]\Big\}$$
$$\times 1/n^2K^4(\frac{1}{2} - J_2(K))^2 \qquad (24)$$

We have represented this result in the figure 4 where we have also represented a numerical calculation of the kurtosis [14].

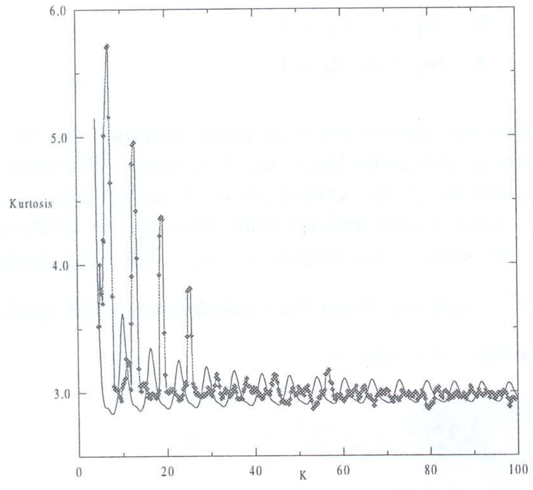

Figure 4: We plot both the analytic and numeric result of the kurtosis versus K, with the solid line without stars is the analytic result and the solid line with stars is the numerical one.

VI. CONCLUSION

We have elaborated a numerical method using mappings for detecting the magnetic field line passage from trapped state to the stochastic one. This method has allowed us to find the Chirikov constant K_c and calculated the couples (K_1, K_2) which are the points where we have a transition partial-global stochasticity. For the study of the magnetic field lines diffusion, we have introduced an empirical model that gives interesting results which are in good qualitative agreement with results found by authors [10,12].
From the analysis of the non-Gaussian dynamics of magnetic field lines, we have shown that they have a Gaussian behavior only for values of the parameter of stochasticity great than 60.

[1] P.H. Rebut. La vie des Sciences. Tome 7, N3 (1990).
[2] M. Hugon; J. T. Mendonça et P. H. Rebut. C.R. Acad.Sci. Paris 308 série II, 1319 (1989).
[3] P.H. Rebut and M. Brustaci. Plasma Physics and Controlled Nuclear Fusion 28,113 (1986).
[4] A.J. Lightenberg and M.A. Lieberman (Springer, NewYork 1989).
[5] J.T. Mendonça. Phys. Fluids B3 January (1991)
[6] D. F. Escande. Phys. Report 121, 165 (1985)
[7] J.A. Krommes and al. J.Plasma Phys. 30,11 (1983).
[8] J.H. Misguich and al.. Plasma Phys. Controlled Fusion 29, 825 (1987).
[9] G. Laval. Phys. Fluids B 5 (3) March 1993.
[10] P. Helander and L. Kjellberg. Phys Plasmas 1 (January 1994).
[11] B. V. Chirikov. Phys. Report 53, 2055 (1979).
[12] G. Zimbardo and P. Veltri. Phys. Rev. E 51 (Februry 1995).
[13] A. Oualyoudine, D. Saifaoui, A. Dezairi and A. Raouak. J. Phys. III France 7, 1045-1061 (1997).
[14] R. Tabet, D. Saifaoui, A. Dezairi and A. Raouak. (Accepted for publication in European Physical Journal).