Bifurcations sets of the Sretensky axial symmetric gyrostat

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In this paper, we perform an adapted Deprit coordinate transformation and we analyse the flow evolution on the phase space for the axial symmetric gyrostat in the Sretensky case .We give a complete description of thegeneric bifurcations of the common level sets of the first integrals. A numerical investigation of these bifurcations is considered.

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1. Introduction

Consider a rigid body moving about a fixed point under the action of uniform gravity and gyroscopic forces, due to symmetric rotors or holes filled with an ideal incompressible fluid. Such a body is usually called heavy gyrostat. Its equation of motion can be written in the form [1]:

$$\begin{cases} \Im \dfrac{d\vec{\omega}}{dt} = (\Im\vec{\omega}+\vec{\lambda})\wedge\vec{\omega}+\mu\,\vec{e}\wedge\vec{r}, \\ \dfrac{d\vec{e}}{dt} = \vec{e}\wedge\vec{\omega} \end{cases} \qquad (1)$$

Where $\vec{\omega}=(p,q,r)$ is the angular velocity, $\vec{e}=(\gamma_1,\gamma_2,\gamma_3)$ is the unit vector along the direction of the gravitational field, $\Im\vec{\omega}=(A\,p,B\,q,C\,r)$ is the angular momentum, $\vec{r}=(x_0,y_0,z_0)$ is the centre of mass (the components of these vectors are referred to the fixed in the body frame, formed by the principal axes of inertia of the body at the fixed point), μ is the mass of the body, A, B, C are the principal moments of inertia, and $\vec{\lambda}=(\lambda_1,\lambda_2,\lambda_3)$ is the gyrostatic momentum (due to the gyroscopic forces).

Denote H the energy of the body:

$$H=\frac{1}{2}\langle\vec{\omega},\Im\vec{\omega}\rangle+\mu\langle\vec{r},\vec{e}\rangle$$

$\langle .,. \rangle$ Is the usual scalar product in R^3.

The vector field (1) restricted to the four dimensional level set:

$$M_{h_1,h_2}=\left\{(\vec{\omega},\vec{e})\in R^6 : H_1=\langle\Im\vec{\omega}+\vec{\lambda},\vec{e}\rangle=h_1;\ H_2=\langle\vec{e},\vec{e}\rangle=h_2=1\right\}$$

is a two degrees of freedom Hamiltonian system. Thus for Liouville complete integrability of the system (1) we need except of the first integrals H_1, H_2 and the Hamiltonian H, an additional integral of motion. Such integral does exist in the following cases:

* $x_0=y_0=z_0=0$, $H_3=\langle\Im\vec{\omega}+\vec{\lambda},\Im\vec{\omega}+\vec{\lambda}\rangle$ (Euler) [2]

* $A=B$, $x_0=y_0=0$, $\lambda_1=\lambda_2=0$, $H_3=r$ (Lagrange) [2]

* $A=B=2C$, $z_0=y_0=0$, $\lambda_1=\lambda_2=0$

$$H_3=\left(C(p^2-q^2)-\mu x_0\gamma_1\right)^2+\left(2Cpq-\mu x_0\gamma_2\right)^2 -4\mu\ x_0\lambda_3 p\gamma_3+2\lambda_3(p^2+q^2)(Cr-\lambda_3)$$

(Yehia) [3]

* $A=B=4C$ and the center of mass lies in the equatorial plane of the inertia ellipsoid. In this case some integrable systems arise only on the zero level of the integral H_1 ($H_1=\langle\Im\vec{\omega}+\vec{\lambda},\vec{e}\rangle=0$)[4]. In particular, the case of Sretensky gyrostat (which is the subject of this paper)corresponding to a gyrostatic momentum along the axis of dynamical symmetry, is characterised by $\vec{\lambda}=(0\ ,0,\lambda=\lambda_3)$. The additional integral is:

$$H_3=(r-\frac{\lambda}{C})(p^2+q^2)-\frac{\mu}{C}\gamma_3(x_0p+y_0q)$$

N.B: The integrable cases of the gyrostat cited above reduce to the well-know integrable cases of rigid body: Euler, Lagrange, Kowalevskaya and Goryatchev-Tchaplygin, for vanishing the gyrostatic momentum λ.

Recently , several integrable cases of motion of the rigid body and the gyrostat under the action of potential and gyroscopic forces have been found ,mainly when these

forces admit a common axis of symmetry fixed in space [5, 6].

In the present paper we describe the phase space topology and obtain the bifurcation sets of the Sretensky gyrostat. For doing that, we separate the Hamiltonian system in a modified Deprit coordinate system constructed for this purpose. This separability implies a description of the topology of the common-level sets of the first integrals (invariant level sets) in order to make up the whole phase space picture. According to Arnold-Liouville theorem, for non-critical values of the first integrals, the invariant level sets are a finite union of two-dimensional tori. As the values of the first integrals vary, we describe all generic bifurcations of the invariant level sets and hence bifurcation of Liouville tori .At the end of the paper, we give a numerical illustration of these bifurcations.

Note that the works of A.T. Fomenko [7] and A.A. Oshemkov [8] in which the topology of invariant level sets of two degrees of freedom integrable Hamiltonian systems (like: rigid body, gyrostat…) is studied in a quite different way. Namely, these works are based on Fomenko's theory of surgery on bifurcation of Liouville tori which consists of the study of iso-energy three dimensional manifold:

$$Q = \{(x, y, p_x, p_y): H = h\}$$

however, in our study we consider the common level sets $A_{\Re}$ of the first integrals:

$$A_{\Re} = \{(x, y, p_x, p_y) \;\; : \;\; H = h, F = f \;\; \}$$

where H and F are respectively the Hamiltonian and the second invariant of the system.

Since the set $A_{\Re}$ is contained in the iso-energy manifold Q, our description gives directly the Liouville tori (and their bifurcations) on which lie the solutions. The most often met iso-energy manifolds Q are: S^3 (the sphere), $\Re P^3$ (the projective space) and $S^2 \times S^1$ (the direct product).

2. Separability of the problem

We shall give an analog to the Deprit variables for the Sretensky gyrostat in order to separate this system. In deed, let us perform a canonical change of variables:

$$(p, q, r; \gamma_1, \gamma_2, \gamma_3) \;\rightarrow\; (l, g, s; L, G, S)$$

The Deprit angles l, g, s (varying from 0 to 2π) are the

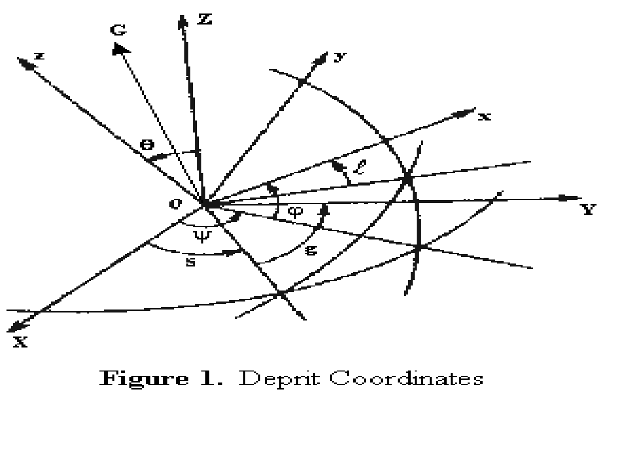

Figure 1. Deprit Coordinates

same as those defined in the classical motion of a rigid body about a fixed point and L, G, S are their canonically conjugate variables [9] .

In our construction, we consider the vector $\vec{G}$ which is the sum of the angular and gyroscopic momentum:

$$\vec{G} = \Im\vec{\omega} + \vec{\lambda}$$

and then:

L is the projection of G on the axis Oz of the gyrostat

G is the magnitude of G

S is the projection of G on the fixed axis OZ. (see figure 1)

N.B: In the classical Deprit coordinates $\vec{G} = \Im\vec{\omega}$.

It can be easy shown that l, g, s and L, G, S are canonically conjugates. Then, the Hamiltonian of the Sretensky gyrostat can be represented with respect to the new variables as:

$$H = \frac{1}{8}\left(G^2 + 3L^2\right) - \lambda L + \mu\left(\frac{L}{G} \sin l \cos g + \sin g \cos l\right) \quad (2)$$

(with the condition: $H_1 = \langle \Im\vec{\omega} + \vec{\lambda}, \vec{e} \rangle = 0$,and without loss of generality $x_0 = 1, y_0 = 0$ and C=1)

The Hamilton-Jacobi equation admits a separation of variables. Indeed, let us perform a canonical change of variables :

$$\begin{aligned} L &= p_x + p_y \qquad & l &= \frac{x+y}{2} \\ G &= p_x - p_y \qquad & g &= \frac{x-y}{2} \end{aligned} \quad (3)$$

The Hamiltonian (2) can be represented in the form (4):

$$H = \frac{p_x^3 - p_y^3}{2(p_x - p_y)} - \lambda\frac{p_x^2 - p_y^2}{(p_x - p_y)} + \mu\left(\frac{p_x}{(p_x - p_y)} \sin x + \frac{p_y}{(p_x - p_y)} \sin y\right)$$

And the corresponding equations of motion are:

$$\begin{aligned}
\dot{x} &= \frac{\partial H}{\partial p_x} = \frac{2p_x + p_y}{2} - \lambda - \mu\frac{p_y}{(p_x - p_y)^2}(\sin x + \sin y) \\
\dot{y} &= \frac{\partial H}{\partial p_y} = \frac{2p_y + p_x}{2} - \lambda + \mu\frac{p_y}{(p_x - p_y)^2}(\sin x + \sin y) \qquad \text{(E)} \\
\dot{p}_x &= -\frac{\partial H}{\partial x} = -\mu\frac{p_x}{(p_x - p_y)} \cos x \\
\dot{p}_y &= -\frac{\partial H}{\partial y} = -\mu\frac{p_y}{(p_x - p_y)^2} \cos x
\end{aligned}$$

where $(\dot{}) = \frac{d}{dt}$ .From (4) we have:

$$\frac{p_x^3}{2} - \lambda p_x^2 + \mu p_x \sin x - hp_x =$$

$$\frac{p_y^3}{2} - \lambda p_y^2 - \mu p_y \sin x - hp_y = const.$$

Denoting this constant F, the additional integral of the system (E) commuting with (4). Then:

$$\frac{p_x^3}{2} - \lambda p_x^2 + \mu p_x \sin x - hp_x = \frac{p_y^3}{2} - \lambda p_y^2 - \mu p_y \sin x - hp_y$$

Let F denote the additional integral of the system (E). Then:

$$F = \frac{p_x^3}{2} - \lambda p_x^2 + \mu p_x \sin x - hp_x$$
$$F = \frac{p_y^3}{2} - \lambda p_y^2 - \mu p_y \sin x - hp_y \qquad (5)$$

In terms of new Deprit variables F has the form:

$$F = \left(G^2 - L^2\right)\left(\frac{L}{8} + \frac{\mu \sin l \cos g}{2G} - \frac{\lambda}{4}\right) \quad (6)$$

After a straight forward computation from the equations (4), (E) and (5), we conclude that p_x and p_y satisfy the system of differential equations:

$$\begin{cases} \dot{p}_x = \pm \dfrac{\sqrt{\phi(p_x)}}{p_x - p_y}, \\ \dot{p}_y = \pm \dfrac{\sqrt{\phi(p_y)}}{p_x - p_y} \end{cases} \qquad (7)$$

where $\phi(z)$ is a polynomial of degree 6 :

$$\phi(z) = \mu^2 z^2 - \left(f - \frac{z^3}{2c} + \lambda z^2 + hz\right)^2 = \phi_1(z)\phi_2(z)$$

$\phi_1(z), \phi_2(z)$ are cubic polynomial defined by:

$$\begin{cases} \phi_1(z) = \dfrac{z^3}{2c} - \lambda z^2 + (\mu - h)z - f \\ \phi_2(z) = -\left(\dfrac{z^3}{2c} - \lambda z^2 + (\mu + h)z - f\right) \end{cases}$$

Thus, the system (E) can be integrated by means of hyperelliptic functions of time on the complexified manifolds:

$$A_C = \left\{\left(x, y, p_x, p_y\right) \in C^4 : H = h, F = f\right\} \subset C^4$$

which is the complexified common level set of the first integrals H and F of the gyrostat.

3. Topological analysis

In order to describe the topology of the real space phase, we consider the system (E) in the real case. Thus, the constants λ, μ , h, f and the flows of H and F will be real .Let $A_{\Re}$ denote the real manifold of the system (E):

$$A_{\Re} = \left\{\left(x, y, p_x, p_y\right) \in \Re^4 \;:\; H = h, F = f\right\} \subset \Re^4$$

From equations (E) and (7) the functions x, y, p_x, p_y depend on the roots of the polynomial ϕ and they are non-degenerate if the discriminant of ϕ is non-zero. Thus we define the bifurcation diagram B as the discriminant locus of the polynomials $\phi_1(z)$ and $\phi_2(z)$ whose coefficients depend on h, f, λ , and μ . (see: [11-14]).B is also the set of the critical values of the energy-momentum mapping:

$$\left(x, y, p_x, p_y\right) \rightarrow (H, F)$$

We find:

$$B = B_1 \cup B_2 = \left\{\left(f, h, \lambda, \mu\right) \in R^4 \;:\; disc(\phi_1(z)) = 0\right\} \cup \left\{\left(f, h, \lambda, \mu\right) \in R^4 \;:\; disc(\phi_2(z)) = 0\right\}$$

$$B_1 : \left\{ (f, h, \lambda, \mu) \in R^4 \;:\; (f = \frac{-8\lambda^3}{27} + \frac{2}{3}(\mu - h)\lambda \pm \frac{2}{27}\sqrt{2(3h - 3\mu + 2\lambda^2)^3} \text{ and } f = 0, with\ h \geq \mu - \frac{\lambda^2}{2} \right\}$$

$$B_2 : \left\{ (f, h, \lambda, \mu) \in R^4 \;:\; (f = \frac{-8\lambda^3}{27} - \frac{2}{3}(\mu + h)\lambda \pm \frac{2}{27}\sqrt{2(3h + 3\mu + 2\lambda^2)^3} \text{ and } f = 0, with\ h \geq -\mu - \frac{\lambda^2}{2} \right\}$$

For a fixed value of μ and varying λ's value (with respect to its dependence of μ), we obtain three types of resultant bifurcation diagram

$B\cap\{\lambda=const,\mu=const\}$ with $\lambda^2<4\mu$.(figure 2a)

$B\cap\{\lambda=const,\mu=const\}$ with $\lambda^2>4\mu$.(figure 2b)

$B\cap\{\lambda=const,\mu=const\}$ with $\lambda^2=4\mu$.(figure 2c)

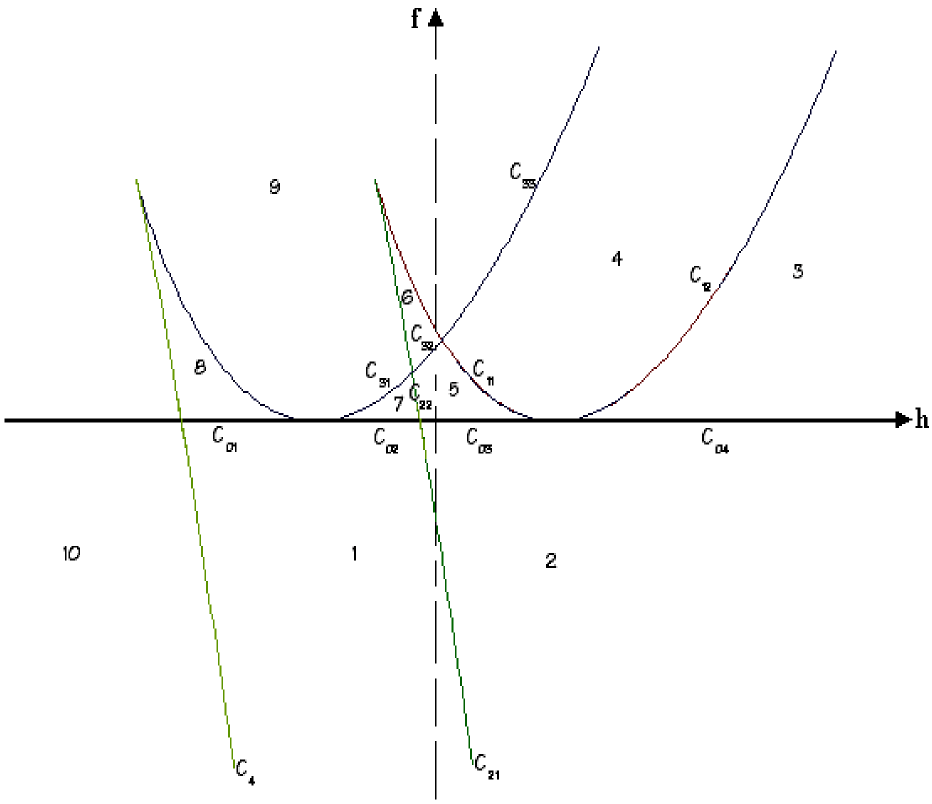

Figure2a: Diagram of Bifurcation $B\cap\{\lambda=const,\mu=const\}$ with $\lambda^2<4\mu$

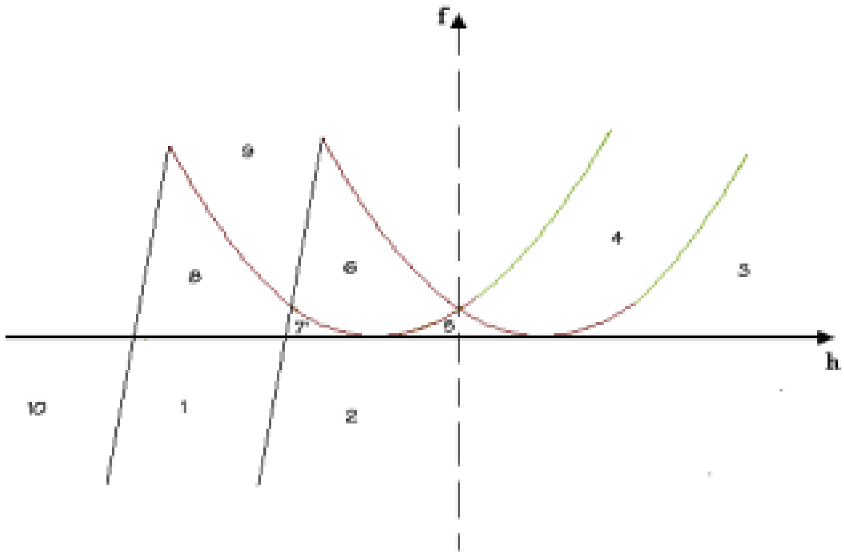

Figure2b: Diagram of Bifurcation $B\cap\{\lambda=const,\mu=const\}$ with $\lambda^2>4\mu$

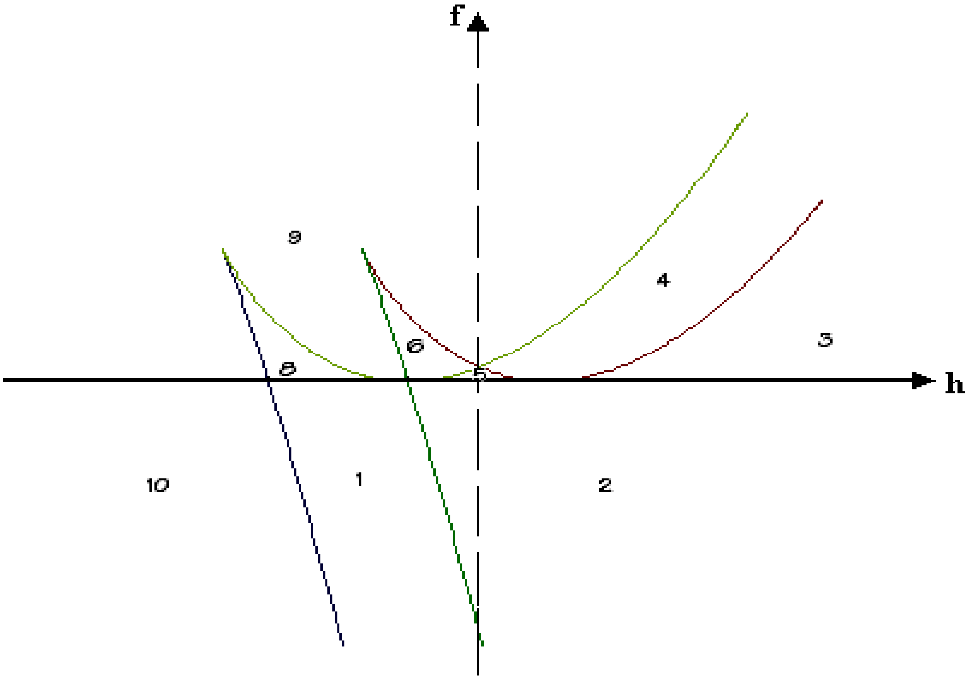

Figure2c : Diagram of Bifurcation $B \cap \{\lambda = const, \mu = const\}$ with $\lambda^2 = 4\mu$

As shown in figures 2, the set $\{\Re^4 \setminus B\} \cap \{\lambda = const, \mu = const\}$ consists of 10 connected components, in each one of them , the level set A_R has the same topological type and it may change only as (f, h, λ, μ) passes through B.

Note that the domains 7 and 7' vanishes when $\lambda^2 = 4\mu$,and when $\lambda = 0$ we have all the results found for the Goryatechev-Tchaplygin top[14].

3.1.Topology of regular level sets

Theorem1:

If $(f, h, \lambda, \mu) \in \{\Re^4 \setminus B\} \cap \{\lambda = const, \mu = const\}$,*the topological type of* $A_\Re$ *is a torus, a disjoint union of two tori or an empty set.*

To prove that, consider the complexified system:

$$A_C = \{(x, y, p_x, p_y) \in C^4 : H = h, F = f\} \subset C^4$$

Denote by Γ the genus-tow hyperelliptic curve such as:

$$\Gamma : \{w^2 = \phi(u)\}$$

We obtain the explicit solutions of the initial problem (5) by solving the Jacobi inversion problem [15].Thus x, y, p_x, p_y can be expressed in terms of hyperelliptic functions living in the Jacobi variety Jac (Γ).

Let's define the natural projection

$$\Pi \ : \ A_{\mathbb{C}} \to Jac(\Gamma)$$

corresponding to the complex conjugation

$$(x, y, p_x, p_y) \to (\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y)$$

The real level set $A_\Re = Re(A_{\mathbb{C}})$ is the set of the fixed points of the complex conjugation on $A_{\mathbb{C}}$. According to Arnold-Liouville theorem, for non-critical values of Hand F the level set $A_\Re$ is a finite union of low dimensional tori ,whose number depends only upon the number and the location of the ovals (admissible ovals) on the Riemann surface associated to the hyperelliptic curve Γ.

To describe $A_\Re$ it is enough to study the projection Π . Thus to determine the ovals of Γ (sets of fixed points), it is enough to study the real roots of the polynomial ϕ (z) for different values of h, f, λ and μ as is shown in table I. Using the formulae (7), the condition of Deprit 's variables $|L| \le G$ and the condition $(x, y, p_x, p_y) \in \Re^4$, we obtain the conditions that assure the existence of the movement:

$$\begin{cases} \phi(p_x) \ge 0 , & \phi(p_y) \ge 0 , \\ p_x \rangle 0 & , \quad p_y \langle 0 \end{cases}$$

These conditions determine the admissible ovals whose projections on the p_x-plane and p_y-plane are given by the intervals Δ_1, Δ_2 (table 2), the direct product of these two intervals gives the topological type of $A_\Re$.

Thus we obtain :

i- $A_\Re$ is a torus in domain 1, 4 , and 7.

ii- $A_\Re$ is a disjoint union of two tori in domain 2, 3 and 5

iii- $A_\Re \sim \varnothing$ in all other domains.

DOMAIN	$\phi_1(z)$ $z = u_i$ $(i = 1,2,3)$	$\phi_2(z)$ $z = v_i$ $(i = 1,2,3)$	$\phi\ (z)$
1	$u_1 < 0$	$v_1 < 0 < v_2 < v_3$	$v_1 < u_1 < 0 < v_2 < v_3$
2	$u_1 < 0 < u_2 < u_3$	$v_1 < 0 < v_2 < v_3$	$v_1 < u_1 < 0 < v_2 < u_2 < u_3 < v_3$
3	$u_1 < u_2 < 0 < u_3$	$v_1 < v_2 < 0 < v_3$	$v_1 < u_1 < v_2 < u_2 < 0 < u_3 < v_3$
4	$u_3 > 0$	$v_1 < v_2 < 0 < v_3$	$v_1 < v_2 < 0 < u_3 < v_3$
5	$0 < u_1 < u_2 < u_3$	$v_1 < v_2 < 0 < v_3$	$v_1 < v_2 < 0 < u_1 < u_2 < u_3 < v_3$
6	$0 < u_1 < u_2 < u_3$	$v_3 > 0$	$0 < u_1 < u_2 < u_3 < v_3$
7	$u_1 > 0$	$v_1 < v_2 < 0 < v_3$	$v_1 < v_2 < 0 < u_1 < v_3$
7’	$0 < u_1 < u_2 < u_3$	$0 < v_1 < v_2 < v_3$	$0 < u_1 < v_1 < v_2 < u_2 < u_3 < v_3$
8	$u_1 > 0$	$0 < v_1 < v_2 < v_3$	$0 < u_1 < v_1 < v_2 < v_3$
9	$u_3 > 0$	$v_3 > 0$	$0 < u_3 < v_3$
10	$u_1 < 0$	$v_1 < 0$	$v_1 < u_1 < 0$

Table 1: Real roots of the polynomials $\phi_1(z), \phi_2(z)$ and $\phi(z)$ for $(h,f,\lambda,\mu) \in \{\Re^4 \setminus B\} \cap \{\lambda = const, \mu = const\}$

DOMAIN	Δ_1	Δ_2	$A_{\Re} = \Delta_1 \times \Delta_2$
1	$[v_2, v_3]$	$[v_1, u_1]$	T
2	$[v_2, u_2] \cup [u_3, v_3]$	$[v_1, u_1]$	2T
3	$[u_3, v_3]$	$[v_1, u_1] \cup [u_2, v_2]$	2T
4	$[u_3, v_3]$	$[v_1, v_2]$	T
5	$[u_1, u_2] \cup [u_3, v_3]$	$[v_1, v_2]$	2T
6	$[u_1, u_2] \cup [u_3, v_3]$	$\varnothing$	$\varnothing$
7	$[u_1, v_3]$	$[v_1, v_2]$	T
7’	$[u_1, v_1] \cup [v_2, u_2] \cup [u_3, v_3]$	$\varnothing$	$\varnothing$
8	$[u_1, v_1] \cup [v_2, v_3]$	$\varnothing$	$\varnothing$
9	$[u_3, v_3]$	$\varnothing$	$\varnothing$
10	$\varnothing$	$[v_1, u_1]$	$\varnothing$

Table 2: Admissible ovals and topological type of for $(h, f, \lambda, \mu) \in \{\Re^4 \setminus B\} \cap \{\lambda = const, \mu = const\}$

3-2.Topology of singular level sets.

In this section, we shall give the description of all generic bifurcations of the topological type of $A_{\Re}$. This means that we have to describe how the topological type of $A_{\Re}$ change as the constants (h, f, λ, μ) passes through the bifurcation diagram B. As the bifurcation sets is obtained from the polynomial $\phi(z)$, when a bifurcation of Liouville tori takes place it is due to the bifurcation of the polynomial $\phi(z)$ roots. Hence the solutions of differential equations (7) of the gyrostat motion change qualitatively and then quantitatively.

We have the following type of bifurcation :

1- A torus shrinks to a circle and then vanishe .
2- A torus splits into two tori (or conversely, two tori glue together).
3- A symmetric bifurcation of one tori into one tori.
4- A symmetric bifurcation of two tori into two tori.
5- Two tori bifurcate into two circles and then vanishes.(it’s equivalent to the first case)

$1 \to 8$ $1 \to 10$ $4 \to 9$ $7 \to 6$	$1 \to 2$ $4 \to 5$ $4 \to 3$ $7 \to 5$	$1 \to 7$	$2 \to 3$ $2 \to 5$	$5 \to 6$ $2 \to 7'$
$T \to \varnothing$	$T \to 2T$	$T \to T$	$2T \to 2T$	$2T \to \varnothing$

Table 3. Generic bifurcations of the level set $A_{\Re}$ passing from domain i to domain j

To prove that, it suffices to look at the bifurcation of roots of the polynomial $\phi(z)$ (table. 4)

Case 1 : Domain1 $\longrightarrow$ C_{01} $\longrightarrow$ Domain8

$T \to S \to \varnothing$

In domain 1: $\Delta_1 = [v_2, v_3]$ *and* $\Delta_2 = [v_1, u_1]$; $A_{\Re}$ = T

In the curve C_{01} Δ_2 reduces to $\{v_1 = u_1 = 0\}$, thus: $A_{\Re}$ =S

In domain 8: $[v_1, u_1] = \varnothing$, the circle vanishes and hence $A_{\Re}$ = $\varnothing$ In a similar way we can prove the other cases (figure3:

Curves	Δ_1	Δ_2	$A_{\Re} = \Delta_1 \times \Delta_2$
C_{11}	$[u_3, v_3] \cup \{u_1 = u_2\}$	$[v_1, v_2]$	$T^2 \cup S^1$
C_{12}	$[u_3, v_3]$	$[v_1, u_1 = u_2] \cup [u_1 = u_2, v_2]$	$S^1 \times (S^1 \vee S^1)$
C_{21}	$[u_1, u_2 = u_3] \cup [u_2 = u_3, v_3]$	$[v_1, v_2]$	$S^1 \times (S^1 \vee S^1)$
C_{22}	$[v_2, u_2 = u_3] \cup [u_2 = u_3, v_3]$	$[v_1, u_1]$	$S^1 \times (S^1 \vee S^1)$
C_{31}	$[u_1, v_3]$	$\{v_1 = v_2\}$	S^1
C_{32}	$[u_1, u_2] \cup [u_3, v_3]$	$\{v_1 = v_2\}$	$2S^1$
C_{33}	$[u_3, v_3]$	$\{v_1 = v_2\}$	S^1
C_4	$\{v_2 = v_3\}$	$[v_1, u_1]$	S^1
C_{01}	$[v_2, v_3]$	$\{v_1 = u_1 = 0\}$	S^1
C_{02}	$[u_1 = v_2 = 0, v_3]$	$[v_1, v_2 = u_1 = 0]$	$S^1 \times (S^1 \wedge S^1)$
$C_{02'}$	$[v_2, u_2] \cup [u_3, v_3]$	$\{v_1 = u_1 = 0\}$	$2S^1$
C_{03}	$[v_2 = u_1 = 0, u_2] \cup [u_3, v_3]$	$[v_1, v_2 = u_1 = 0]$	$T^2 \cup (S^1 \times (S^1 \wedge S^1))$
C_{04}	$\{v_2 = u_2 = 0\} \cup [u_3, v_3]$	$[v_1, u_1]$	$T^2 \cup S^1$

Table4: Topological type of $A_{\Re}$ on the diagram B.Where $(S^1 \vee S^1)$ is two-coplanar circles having one common point (the eight figure) and $(S^1 \wedge S^1)$ is the two no-coplanar circles having one common point

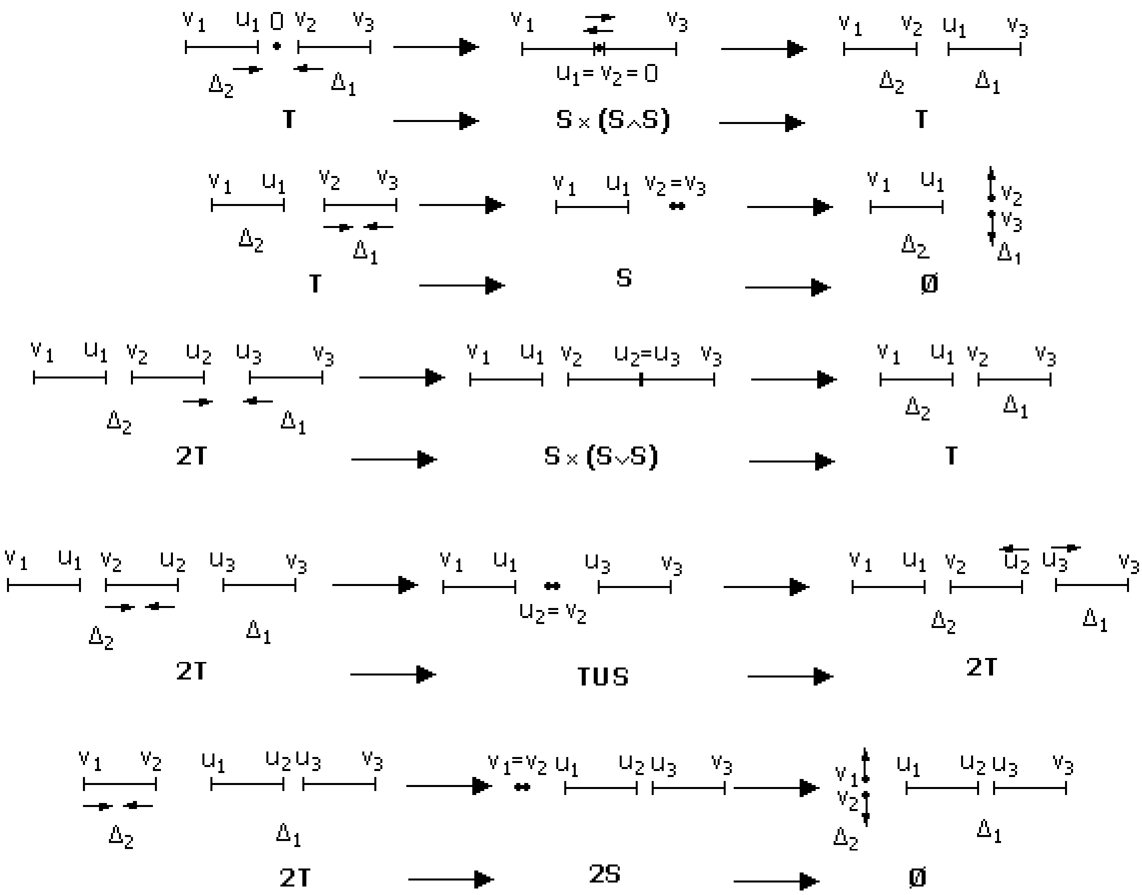

Figure3. Correspondence between bifurcations of Liouville tori and polynomials roots.

4. Numerical investigation

Considering the state space (l,g,L,G); the global dynamics of the sequence of bifurcations of Liouville tori of Stretensky gyrostat may be viewed via the surfaces of section map (poincaré sections). For fixed values of energy h, as f varies and vice-versa, the Liouville tori contained in the level set {H=h; F=f} change their topological type. The corresponding surfaces of section map are shown in figure 4 and figure 5. This map is constructed by integrating the Hamiltonian equations (E) and computing their intersections with a surface of section map using Henon's algorithm [16] .All surfaces of section are contained in the windows:

$$\left\{ (\ell, \frac{L}{G}) : 0 \le \ell \le 2\pi ; -1 \le \frac{L}{G} \le 1 \right\}$$

and are obtained for h=3 varying f as shown in Figure 4 (a, b, c, d, e, f ,g ,h ,i). For the bifurcation T→T, we have choose h=-0.16 and f=0 (a (h ,f) point on the curve C02)as shown in Figure 5.

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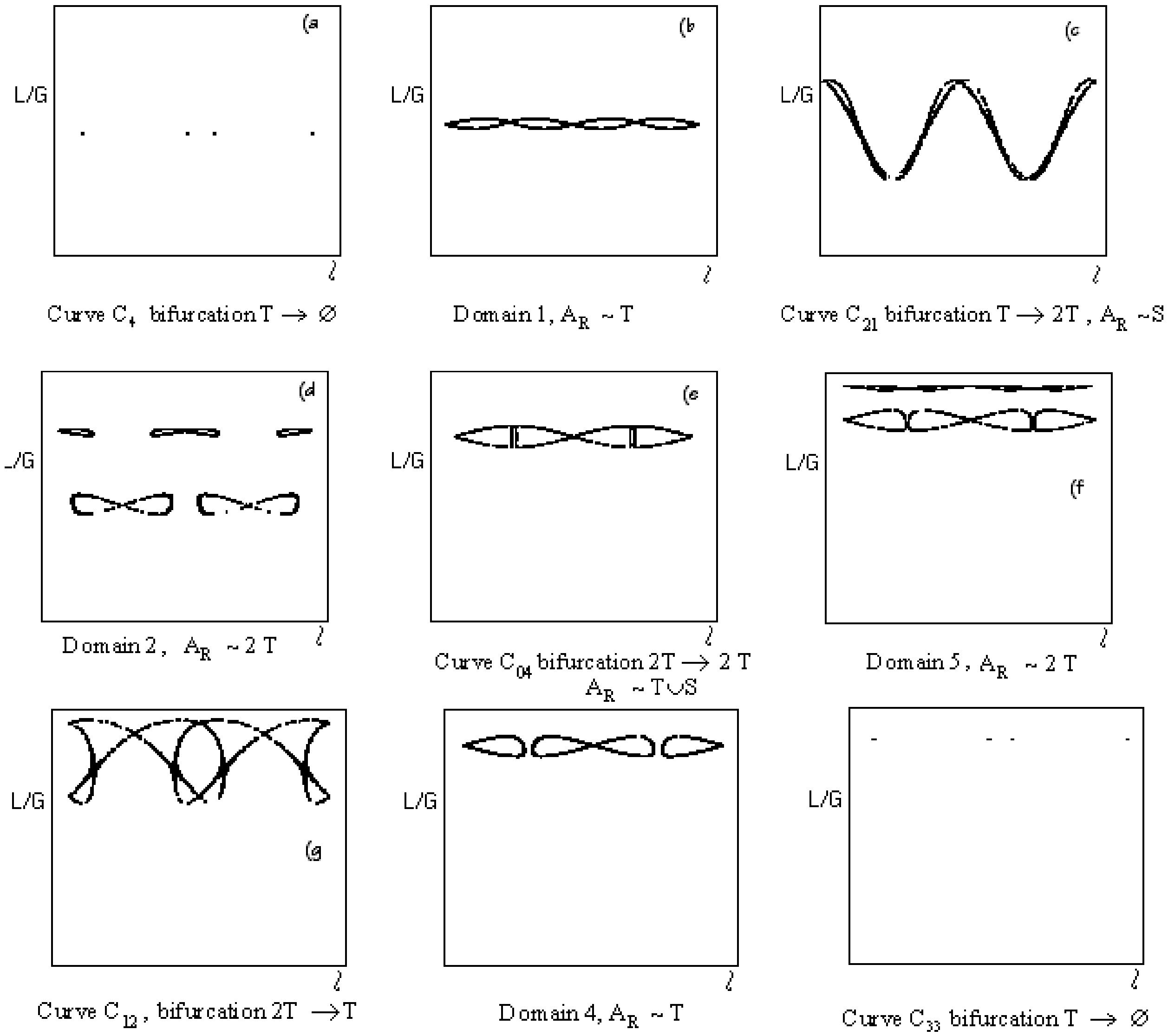

Figure 4. Surface of section map for h=3,varying f .
Passing from domain 1$\rightarrow$2$\rightarrow$3$\rightarrow$4$\rightarrow$9

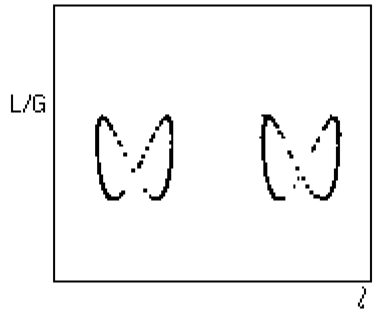

Figure 5. Curve C_{f0} bifurcation T $\rightarrow$ T

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